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Towards a Fully Nonlinear Sharp Sobolev Trace Inequality

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. We classify local minimizers of $\int\sigma_2+\oint H_2$ among all conformally flat metrics in the Euclidean $(n+1)$-ball, $n=4$ or $n=5$, for which the boundary has unit volume, subject to an ellipticity assumption. We also classify local minimizers of the analogous functional in the critical dimension $n+1=4$. If minimizers exist, this implies a fully nonlinear sharp Sobolev trace inequality. Our proof is an adaptation of the Frank–Lieb proof of the sharp Sobolev inequality, and in particular does not rely on symmetrization or Obata-type arguments.

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1 Introduction

The first sharp Sobolev trace inequality was proven by Escobar [21]. In geometric terms, he showed that if g is any conformally flat metric on the Euclidean ball $B^{n+1}\subset\mathbb{R}^{n+1}$, $n>1$, of radius one, then

$$\frac{1}{2n}\int_{B^{n+1}} R^g\,\mathrm{dvol}_g+\oint_{S^n} H^g\,\mathrm{dvol}_{\iota^*g}\geq\omega_n^{\frac{1}{n}}\mathrm{Vol}_{\iota^*g}(S^n)^{\frac{n-1}{n}},\tag{1.1}$$

where ω_n is the volume of the standard n-sphere, $\iota\colon S^n\to B^{n+1}$ is the inclusion of $S^n=\partial B^{n+1}$, and H^g is the mean curvature of S^n induced by g, with the convention that S^n has mean curvature 1 with respect to the standard metric. Moreover, he showed that equality

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holds in (1.1) if and only if g is flat. His proof relies on an Obata-type argument which classifies all conformally flat, scalar flat metrics g on the ball for which the boundary has constant mean curvature. The inequality (1.1) plays a crucial role in studying a version of the boundary Yamabe problem; see [2, 22, 31–33] and references therein.

In analytic terms, Eq. (1.1) states that

$$\int_{B^{n+1}} vL_2v + \oint_{S^n} vB_1v \geq \frac{n-1}{2}\omega_n^{\frac{1}{n}}\left(\oint_{S^n}|v|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} \tag{1.2}$$

for all $v\in W^{1,2}(B^{n+1})$, where $L_2=-\Delta$ is the conformal Laplacian, $B_1=\partial_r+\frac{n-1}{2}$ is the conformal Robin operator [19, 21], and all integrals are taken with respect to the Riemannian volume element of the Euclidean metric on B^{n+1} or the induced metric on S^n, as appropriate. The link between Eq. (1.1) and Eq. (1.2) is obtained by setting $g=v^{\frac{4}{n-1}}g$. Moreover, equality holds if (1.2) if and only if

$$v(x)=a|rx-\xi_0|^{1-n}$$

for constants $a\in\mathbb{R}$ and $r\in[0,1)$ and a point $\xi_0\in S^n$. The inequalities (1.1) and (1.2) are equivalent due to the conformal covariance of L_2 and B_1. Other proofs of (1.2) which exploit conformal covariance and the linearity of L_2 and B_1 are known; e.g. [5, 10].

Given $k\in\mathbb{N}$, Viaclovsky [38] defined the *σ_k-curvature* of a Riemannian manifold (X^{n+1}, g) as the k-th elementary symmetric function of the eigenvalues of the *Schouten tensor* $P:=\frac{1}{n-1}\left(\mathrm{Ric}-\frac{R}{2n}g\right)$. For example, $\sigma_1=\frac{1}{2n}R$. Let $[g_0]$ be the set of metrics conformal to g_0. When written in terms of a fixed background metric g_0, the equation $\sigma_k^g=f$, $g\in[g_0]$, is a second-order fully nonlinear PDE which is elliptic in the positive k-cone; i.e. it is elliptic if $\sigma_j^g>0$ for $1\leq j\leq k$; see [38]. On closed manifolds, the equation $\sigma_k^g=1$ is variational if and only if $k\leq 2$ or g is locally conformally flat [8].

Initial studies of the σ_k-curvature involved constructing minimizers of the total σ_k-curvature functional among all volume-normalized metrics in the positive k-cone (e.g. [29, 30, 37]). In the critical case of dimension four, Chang, Gursky and Yang [13] noted that one could instead work in the positive 1-cone provided the total σ_2-curvature was positive. Later studies (e.g. [26, 27, 36]) generalized this to show that one can minimize in the positive $(k-1)$-cone under a suitable integral assumption. For example, combining results of Guan and Wang [28] and Ge and Wang [26] yields sharp fully nonlinear Sobolev inequalities of closed n-spheres, $n>4$, stated in terms of the σ_2-curvature and the positive 1-cone. Note that Obata's argument generalizes to prove that any conformally flat metric of constant σ_k-curvature on the sphere has constant sectional curvature, subject to the above ellipticity condition [14, 38].

Given $k\in\mathbb{N}$, S. Chen [18] defined the *H_k-curvature* of the boundary of a Riemannian manifold (X^{n+1},g) in terms of elementary symmetric functions of the Schouten tensor of the interior and the second fundamental form of the boundary. The key points are that H_1 is the mean curvature; H_k^g, $g\in[g_0]$, when written in terms of a fixed boundary metric