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Submanifolds in Cauchy Riemann Geometry

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Dedicated to Professors Sun-Yung Alice Chang and Paul C.-P. Yang on their 70th birthdays

Abstract. In this paper I would like to make a report on the results about hypersurfaces in the Heisenberg group and invariant curves and surfaces in CR geometry. The results are contained in the papers [8, 9, 16] and [14]. Besides, I would also report on the results about the strong maximum principle for a class of mean curvature type operators in [10].

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1 Introduction

We start with a summary of the results. In [8], we defined a notion of umbilicity for hypersurfaces in the Heisenberg group H_n with $n \geq 2$ (see Section 2 for some basic material about H_n). We classified umbilic hypersurfaces in some cases, and proved that *Pansu spheres* are the only umbilic spheres with positive constant p(or horizontal)-mean curvature in H_n up to Heisenberg translations. In [9] we studied immersed, connected, umbilic hypersurfaces in the Heisenberg group H_n with $n \geq 2$. We showed that such hypersurfaces, if closed, must be rotationally invariant up to Heisenberg translations. Moreover, we proved that, among others, Pansu spheres are the only such spheres with positive constant sigma-k curvature up to Heisenberg translations.

In [10], we studied the strong maximum principle for horizontal (p-) mean curvature operator and p-(sub)laplacian operator on subriemannian manifolds including, in particular, Heisenberg groups and Heisenberg cylinders. Under a certain Hormander type

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condition on vector fields, we showed that the strong maximum principle holds in higher dimensions for two cases: (a) the touching point is nonsingular; (b) the touching point is an isolated singular point for one of comparison functions. For a background sub-riemannian manifold with local symmetry of isometric translations, we have the strong maximum principle for associated graphs which include, among others, intrinsic graphs with constant horizontal (p-) mean curvature. As applications, we showed a rigidity result of horizontal (p-) minimal hypersurfaces in any higher dimensional Heisenberg cylinder and a pseudo-halfspace theorem for any Heisenberg group.

In the study of CR invariant curves (called chains) and surfaces (Willmore-type surfaces), we have published the following two papers [14] and [16]. In [14], with the help of a generalization of the Fermat principle in general relativity, we showed that chains in CR geometry are geodesics of a certain Kropina metric in Finsler geometry, constructed from the CR structure. We studied the projective equivalence of Kropina metrics and showed that if the kernel distributions of the corresponding 1-forms are non-integrable, then two projectively equivalent metrics are trivially projectively equivalent. As an application, we showed that sufficiently many chains determine the CR structure up to conjugacy, generalizing and reproving the main result of [6]. The correspondence between geodesics of the Kropina metric and chains allowed us to use the methods of metric geometry and the calculus of variations to study chains. We used these methods to re-prove a result of Howard Jacobowitz that locally any two points of a strictly pseudoconvex CR manifolds can be joined by a chain. Finally, we generalized this result to the global setting by showing that any two points of a connected compact strictly pseudoconvex CR manifold which admits a pseudo-Einstein contact form with positive Tanaka-Webster scalar curvature can be joined by a chain.

In [16], we expressed two CR invariant surface area elements in terms of quantities in pseudohermitian geometry. We deduced the Euler-Lagrange equations of the associated energy functionals. Many solutions were given and discussed. In relation to the singular CR Yamabe problem, we showed that one of the energy functionals appears as the coefficient (up to a constant multiple) of the log term in the associated volume renormalization.

2 Umbilicity and characterization of Pansu spheres

In classical differential geometry, we have the notion of umbilicity for a point in a hypersurface of the Euclidean space R^n. A connected, closed umbilic hypersurface of R^n (i.e., all the points are umbilic) is shown to be a sphere. On the other hand, we have the Alexandrov theorem which says that a closed (compact with no boundary) hypersurface of positive constant mean curvature in R^n must be a sphere. The original proof of Alexandrov's theorem is based on a reflection principle. Reflect the hypersurface S across a hyperplane P. Move P parallely until the reflected hypersurface touches the original hypersurface S. The reflected hypersurface must coincide with S by the strong maximum principle. Analytic proofs of Alexandrov's theorem were given much later. In