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The Fundamental and Rigidity Theorems for Pseudohermitian Submanifolds in the Heisenberg Groups

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. In this paper, we study some basic geometric properties of pseudohermitian submanifolds of the Heisenberg groups. In particular, we obtain the uniqueness and existence theorems, and some rigidity theorems.

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Key words: Motion equations, structure equations, Darboux frame, Darboux derivative.

1 Introduction

In this paper, for $m \leq n$, we specify the ranges of indices as follows

$$\begin{aligned} &1 \leq \alpha,\beta,\gamma,\sigma,\rho,\cdots \leq n, \quad 1 \leq j,k,l,\cdots \leq m,\\ &m+1 \leq a,b,c,\cdots \leq n, \quad 1 \leq A,B,C,\cdots \leq 2n. \end{aligned}$$

1.1 The Heisenberg groups

The origin of pseudohermitian geometry came from the construction of a pseudohermitian connection, independently by N. Tanaka [15] and S. Webster [16]. In this paper, the Heisenberg group is a pseudohermitian manifold and it plays the role of the model in pseudohermitian geometry. That is, any pseudohermitian manifold with vanishing curvature and torsion locally is part of the Heisenberg group. Let H_n be the Heisenberg group, with coordinates (x_β,y_β,t). The group multiplication is defined by

$$(x,y,t)\circ(x',y',t') = (x+x',y+y',t+t'+yx'-xy').$$

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The associated standard CR structure J and contact form Θ are defined respectively by

$$J\mathring{e}_\beta = \mathring{e}_{n+\beta}, \qquad J\mathring{e}_{n+\beta} = -\mathring{e}_\beta,$$
$$\Theta = dt + \sum_{\beta=1}^{n} x_\beta dy_\beta - y_\beta dx_\beta,$$

where

$$\mathring{e}_\beta = \frac{\partial}{\partial x_\beta} + y_\beta \frac{\partial}{\partial t}, \ \mathring{e}_{n+\beta} = \frac{\partial}{\partial y_\beta} + x_\beta \frac{\partial}{\partial t}.$$

The contact bundle is $\xi = \ker\Theta$. We refer the reader to [2, 3, 5] for the details about the Heisenberg groups, and to [6, 11, 12, 15, 16] for pseudohermitian geometry.

The symmetry group $PSH(n)$ of H_n is the group consisting of all pseudohermitian transformations. Left translations L_p are symmetries. Another kind of examples are a rotation Φ_R around the t-axis which is defined by

$$\Phi_R \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix},$$

where $R = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(2n)$. In [5], we showed that each symmetry $\Phi \in PSH(n)$ has the unique decomposition $\Phi = L_p \circ \Phi_R$, for some $p \in H_n$ and $R \in SO(2n)$. Since the action of $PSH(n)$ on H_n is transitive, the associated geometry is a kind of Klein geometry. The corresponding Cartan geometry is just pseudohermitian geometry.

1.2 Pseudohermitian submanifolds

We now give the definition of pseudohermitian submanifold.

Definition 1.1. *A $(2m+1)$-dimensional pseudohermitian manifold $(M,\hat{J},\hat{\theta})$ is called a pseudohermitian*† *submanifold of H_n, $1 \le m \le n$, if*

- $\hat{\xi} = TM \cap \xi$;
- $\hat{J} = J|_{\hat{\xi}}$;
- $\hat{\theta} = \Theta|_M$,

where $\hat{\xi} = \ker\hat{\theta}$ is the contact structure on M. The number m is called the CR dimension of M.

Example 1.1. Suppose $M \hookrightarrow H_n$ is an embedded submanifold with CR dimension $n-1$. Then it is not hard to see that

†In [6], S. Dragomir and G. Tomassini call it *isopseudo-hermitian*, instead of *pseudohermitian*.