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Solutions to the σ_k-Loewner-Nirenberg Problem on Annuli are Locally Lipschitz and Not Differentiable

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Abstract. We show for $k\geq 2$ that the locally Lipschitz viscosity solution to the σ_k-Loewner-Nirenberg problem on a given annulus $\{a<|x|<b\}$ is $C^{1,\frac{1}{k}}_{\rm loc}$ in each of $\{a<|x|\leq\sqrt{ab}\}$ and $\{\sqrt{ab}\leq|x|<b\}$ and has a jump in radial derivative across $|x|=\sqrt{ab}$. Furthermore, the solution is not $C^{1,\gamma}_{\rm loc}$ for any $\gamma>\frac{1}{k}$. Optimal regularity for solutions to the σ_k-Yamabe problem on annuli with finite constant boundary values is also established.

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Key words: σ_k-Loewner-Nirenberg problem, σ_k-Yamabe problem, viscosity solution, regularity, conformal invariance.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n\geq 3$. For a positive C^2 function u defined on an open subset of $\mathbb{R}^n$, let A^u denote its conformal Hessian, namely

$$A^u=-\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u+\frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u\otimes\nabla u-\frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I, \tag{1.1}$$

and let $\lambda(-A^u)$ denote the eigenvalues of $-A^u$. Note that A^u, considered as a $(0,2)$ tensor, is the Schouten curvature tensor of the metric $u^{\frac{4}{n-2}}\mathring{g}$, where $\mathring{g}$ is the Euclidean metric.

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For $1\le k\le n$, let $\sigma_k:\mathbb{R}^n\to\mathbb{R}$ denote k-th elementary symmetric function

$$\sigma_k(\lambda)=\sum_{i_1<\ldots<i_k}\lambda_{i_1}\cdots\lambda_{i_k},$$

and let Γ_k denote the cone $\Gamma_k=\{\lambda=(\lambda_1,\ldots,\lambda_n):\sigma_1(\lambda)>0,\cdots,\sigma_k(\lambda)>0\}$.

In [7, Theorem 1.1], it was shown that the σ_k-Loewner-Nirenberg problem

$$\sigma_k(\lambda(-A^u))=2^{-k}\begin{pmatrix} n\\ k\end{pmatrix},\qquad \lambda(-A^u)\in\Gamma_k,\qquad u>0\qquad \text{in }\Omega, \tag{1.2}$$

$$u(x)\to\infty \text{ as dist }(x,\partial\Omega)\to 0 \tag{1.3}$$

has a unique continuous viscosity solution u and such u belongs to $C^{0,1}_{\rm loc}(\Omega)$. Furthermore, u satisfies

$$\lim_{d(x,\partial\Omega)\to 0} u(x)d(x,\partial\Omega)^{-\frac{n-2}{2}}=C(n,k)\in(0,\infty). \tag{1.4}$$

Eq. (1.2) is a fully nonlinear elliptic equation of the kind considered by Caffarelli, Nirenberg and Spruck [3]. We recall the following definition of viscosity solutions which follows Li [20, Definitions 1.1 and 1.1′] (see also [19]) where viscosity solutions were first considered in the study of nonlinear Yamabe problems.

Let

$$\overline{S}_k:=\left\{\lambda\in\Gamma_k\,\middle|\,\sigma_k(\lambda)\ge 2^{-k}\begin{pmatrix} n\\ k\end{pmatrix}\right\}, \tag{1.5}$$

$$\underline{S}_k:=\mathbb{R}^n\setminus\left\{\lambda\in\Gamma_k\,\middle|\,\sigma_k(\lambda)> 2^{-k}\begin{pmatrix} n\\ k\end{pmatrix}\right\}. \tag{1.6}$$

Definition 1.1. *Let $\Omega\subset\mathbb{R}^n$ be an open set and $1\le k\le n$. We say that an upper semi-continuous (a lower semi-continuous) function $u:\Omega\to(0,\infty)$ is a sub-solution (super-solution) to* (1.2) *in the viscosity sense, if for any $x_0\in\Omega$, $\varphi\in C^2(\Omega)$ satisfying $(u-\varphi)(x_0)=0$ and $u-\varphi\le 0$ $(u-\varphi\ge 0)$ near x_0, there holds*

$$\lambda\big(-A^\varphi(x_0)\big)\in\overline{S}_k,\qquad \big(\lambda\big(-A^\varphi(x_0)\big)\in\underline{S}_k, \textit{respectively}\big).$$

We say that a positive function $u\in C^0(\Omega)$ satisfies (1.2) *in the viscosity sense if it is both a sub- and a super-solution to* (1.2) *in the viscosity sense.*

Eq. (1.2) satisfies the following comparison principle, which is a consequence of the principle of propagation of touching points [23, Theorem 3.2]: If v and w are viscosity sub-solution and super-solution of (1.2) and if $v\le w$ near $\partial\Omega$, then $v\le w$ in Ω; see [7, Proposition 2.2]. The above mentioned uniqueness result for (1.2)-(1.3) is a consequence of this comparison principle and the boundary estimate (1.4).

In the rest of this introduction, we assume that Ω is an annulus $\{a<|x|<b\}\subset\mathbb{R}^n$ with $0<a<b<\infty$, unless otherwise stated. C^2 radially symmetric solutions to (1.2) were