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Light Refraction is Nonlinear Optimisation

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. In this paper, we show that the near field light refraction can be formulated as a nonlinear optimisation problem, both for a point light source and for a parallel light source.

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1 Introduction

Geometric optics such as light reflection and refraction problems have attracted much attention in recent years, due to various practical applications. For instance, an exciting connection with the research of artificial intelligence has been discovered in [18]. For light reflection problems, in [24] it was showed that the far field case is an optimal transportation problem, and thus a linear optimisation problem. Later on, in [19] it was showed that the near field case can be formulated into a nonlinear optimisation problem. The various regularity results for reflection problems have been obtained by many people. We refer the readers to [1, 5, 14, 21, 23] and references therein. In particular, the work of [23] inspired the discovery of Ma-Trudinger-Wang condition in optimal transportation [22], see also [2, 3] for some recent results on associated Monge-Ampère equations. The regularity theory was then extended to more general Monge-Ampère type equations and generated Jacobian equations as well, see, e.g., [6, 11, 12]. Similar to light reflection, light refraction problems are also divided into the far field case and the near field case

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depending on the location of the target. The far field refraction problem has been studied by Gutiérrez-Huang [7] using mass transport, and Karakhanyan [13] for the regularity of weak solutions.

In this paper, we shall focus on the near field case and formulate it into a class of nonlinear optimisation problems. Depending on the type of light source, the refraction system could be of point light source or parallel light source. The near field refraction problem with a point source can be described as follows: Suppose the light emits from the origin O surrounded by medium I with positive intensity f on $\Omega\subset\mathbb{S}^n$. There is a surface $\mathcal{R}$, separates two homogeneous and isotropic media I and II, such that all rays refracted by $\mathcal{R}$ into medium II illuminate a target hypersurface Ω^* in $\mathbb{R}^{n+1}$ with positive intensity g on Ω^*. Let n_1,n_2 be the indices of refraction of media I, II, respectively, and $\kappa=n_2/n_1$. When $\kappa<1$, the refracted rays tend to bent away from the normal, while when $\kappa>1$, the refracted rays tend to bent towards the normal.

In the case of $\kappa<1$, let $(u,v)\in C(\Omega)\times C(\Omega^*)$ and define the constraint function ϕ: $\Omega\times\Omega^*\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$\phi(x,y,u,v)=u+v+\log\left(\frac{1-\kappa^2}{1-e^v(\kappa^2 x\cdot y+\sqrt{\Delta(x,y,v)})}\right), \tag{1.1}$$

where

$$\Delta(x,y,v)=\kappa^2e^{-2v}-2\kappa^2(x\cdot y)e^{-v}+\kappa^4(x\cdot y)^2+\kappa^2(1-\kappa^2)|y|^2. \tag{1.2}$$

As in [8], we assume the following physical condition on Ω and Ω^* that ensures the quantity $\Delta(x,y,v)$ is positive and increasing in v for all $x\in\Omega$, $y\in\Omega^*$, and ϕ is well defined:

(R_1) There exists a constant τ with $0<\tau<1-\kappa$ such that $x\cdot y\geq(\kappa+\tau)|y|$ for all $x\in\overline{\Omega}$ and all $y\in\overline{\Omega^*}$.

Denote the restriction set K by

$$K:=\Big\{(u,v)\in C(\Omega)\times C(\Omega^*):\phi(x,y,u(x),v(y))\leq 0 \text{ in } \Omega\times\Omega^*\Big\}. \tag{1.3}$$

Assume that the intensity functions f,g satisfy the energy conservation condition

$$\int_\Omega f=\int_{\Omega^*}g, \tag{1.4}$$

and f,g are bounded from below and above, namely for some positive constant c_0,

$$c_0\leq f\leq c_0^{-1} \text{ in } \Omega,\quad c_0\leq g\leq c_0^{-1} \text{ in } \Omega^*. \tag{1.5}$$

Definition 1.1. *A mapping $S:\Omega\to\Omega^*$ is called* measure preserving, *and denoted by $S_\#f=g$, if for any $h\in C(\overline{\Omega^*})$,*

$$\int_\Omega h(Sx)f(x)dx=\int_{\Omega^*}h(y)g(y)dy. \tag{1.6}$$