J. Math. Study
doi: 10.4208/jms.v54n2.21.03

Vol. **54**, No. 2, pp. 164-170
June 2021

Regularity for Almost Convex Viscosity Solutions of the Sigma-2 Equation

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Received December 4, 2019; Accepted March 3, 2020;
Published online January 27, 2021.

Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. We establish interior regularity for almost convex viscosity solutions of the sigma-2 equation.

AMS subject classifications: 35D10, 35J60, 35J99

Key words: Regularity, viscosity solution, fully nonlinear, elliptic partial differential equation.

1 Introduction

In this paper, we establish regularity for almost convex viscosity solutions of the σ_2 equation

$$F\left(D^2u\right)=\sigma_2(\lambda)-1=\sum_{1\le i<j\le n}\lambda_i\lambda_j-1=0, \tag{1.1}$$

where $\lambda_i's$ are the eigenvalues of the Hessian D^2u.

Fully nonlinear equation (1.1) is the quadratic analogue of the Laplace equation $\sigma_1=\Delta u$ and the Monge-Ampère equation $\sigma_n=\det D^2u$. In dimension three, $\sigma_2=1$ if and only if $\sum_{i=1}^3\arctan\lambda_i=\pm\pi/2$, which is the special Lagrangian equation from calibrated geometry. The equation $\sigma_2(\kappa)=1$ prescribes the scalar curvature of a Euclidean hypersurface $(x,u(x))$ with principal curvatures $(\kappa_1,\cdots,\kappa_n)=\kappa$. Complex σ_2-type equations arise from the Strominger system in string theory, and the σ_2 function of the Schouten tensor arises in conformal geometry.

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Theorem 1.1. *Let u be a semiconvex viscosity solution of $\sigma_2\left(D^2u\right)=1$ on $B_1(0)\subset\mathbb{R}^n$ with $\Delta u>0$ and $D^2u\geq(\delta-K)I$ for some $\delta>0$ and $K=[n(n-1)/2]^{-1/2}$. Then u is analytic on $B_1(0)$ and has the effective Hessian bound*

$$\left|D^2u(0)\right|\leq C(n)\exp\left[C(n)\operatorname*{osc}_{B_1(0)}u\right]^2.$$

One quick consequence is that every entire almost convex (such as in Theorem 1.1) viscosity solution of (1.1) is a quadratic function; the smooth case was done in [5]. Recall the classic rigidity results for the equations $\triangle u=1$ and $\det D^2u=1$: every entire convex viscosity solution is quadratic. Our result shows that if a singular viscosity solution of (1.1) exists, then it is not convex, or even almost convex.

The interior regularity for (1.1) in general dimensions is a longstanding problem. Sixty years ago, Heinze [8] achieved a priori estimates and regularity for two dimensional Monge-Ampère type equations including (1.1) with $n=2$ by two dimensional techniques. More than ten years ago, a priori estimates and regularity for (1.1) with $n=3$ were obtained via the minimal surface structure of equation (1.1) in the joint work with Warren [17]. Along this "integral" way, Qiu [12] has proved a priori Hessian estimates–then regularity follows–for three dimensional (1.1) with $C^{1,1}$ variable right hand side, and even with left hand side λ replaced by the principal curvatures κ. Hessian estimates for convex smooth solutions of general quadratic Hessian equations in general dimensions have been obtained via a pointwise approach by Guan and Qiu [7]. Hessian estimates for almost convex smooth solutions of (1.1) in general dimensions have been derived by a compactness argument in [10], and recently for semiconvex smooth solutions in [13] using new mean value and Jacobi inequalities.

In contrast, there are Pogorelov-like singular convex viscosity solutions of the symmetric Hessian equations $\sigma_k(\lambda)=1$ with $k\geq 3$ in dimension $n\geq 3$. Under a strict k-convexity assumption on weak/viscosity solutions of $\sigma_k(\lambda)=1$, a priori Hessian estimates and then regularity were obtained by Pogorelov [11] and Chou-Wang [6], for $k=n$ and $2\leq k<n$ respectively, using Pogorelov's pointwise technique. Lastly, we also mention a priori gradient estimates by Trudinger [14] and a priori Hessian estimates for solutions of σ_k as well as σ_k/σ_n equations in terms of certain integrals of the Hessian by Urbas [15,16], Bao-Chen-Guan-Ji [1].

Extending the above a priori estimates to regularity statements about viscosity solutions of (1.1) is more subtle. In dimensions two and three, one can smoothly solve the Dirichlet problem on interior balls with smoothly approximated boundary data; a limiting procedure combined with the a priori estimates then yields the desired interior regularity for the viscosity solution. However, for dimension $n\geq 4$, a priori estimates are not known for general solutions of (1.1). Because the smooth approximations may not satisfy the convexity constraints, we cannot invoke the available a priori estimates while taking the limit and deduce interior regularity.

We circumvent this difficulty using the improved regularity properties of the equation for the Legendre-Lewy transform $\bar{u}(\bar{x})$ found in [5]. By the analytical definition of the