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Regularity of Weak Solutions to a Class of Complex Hessian Equations on Kähler Manifolds

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Abstract. We prove the smoothness of weak solutions to a class of complex Hessian equations on closed Kähler manifolds, by use of the smoothing property of the corresponding gradient flow.

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Key words: Complex Hessian equation, regularity of weak solutions, pluripotential theory, parabolic flows.

1 Introduction

Let (M,ω) be a compact Kähler manifold of dimension $n\geq 2$. For convenience, we write $\omega=\sqrt{-1}g_{i\bar{j}}dz^i\wedge d\bar{z}^j$ in a local coordinate chart. Let u be a smooth function of M. We denote $\omega_u=\omega+\sqrt{-1}\partial\bar{\partial}u$. Locally it can be written as $\sqrt{-1}(g_{i\bar{j}}+u_{i\bar{j}})dz^i\wedge d\bar{z}^j$. Then we consider the complex Hessian equations of the form

$$(\omega+dd^c u)^k\wedge\omega^{n-k}=f(x,u)\omega^n, \tag{1.1}$$

where $f(x,z)\in C^{2,\alpha}(M\times\mathbb{R})$. When $k=1$, it is a quasilinear equation; when $k=n$, it is the complex Monge-Ampére equation, which has been studied extensively in the literature.

For the non-degenerate case of (1.1), when $f(x,u)=f(x)$ and $0<f(x)\in C^{2,\alpha}(M)$, the existence of smooth solutions have been studied by Hou [13], Hou, Ma and Wu [12], and finally solved by Dinew and Kołodziej. However, like the similar case in complex Monge-Ampère equations, when the right hand side $f(x,u)$ depends on the solution u, the existence will rely on the condition $\frac{\partial f}{\partial z}(x,z)>0$.

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The pluri-potential theory for complex Hessian equations have also attracted many attentions, and it was developed in [2, 8, 16, 17] etc. In [16], Lu has considered the weak solution to (1.1) and obtained its existence under some integral conditions. He also studied the approximation properties and the stability of this equation. Furthermore, in [18], Nguyen has studied the Hölder continuity of the solution when $f(x,u)=f(x)\in L^p(M)$ with varying choice of p.

In this paper, we are going to study the regularity of equation (1.1) for smooth function $f(x,z)$ under the assumption that the equation (1.1) has a weak solution. Our method is based on the arguments in [19]. As an application, we show the regularity of the solutions of the equation which has been studied by Lu in [16]. We also improved our previous result in [20].

Our main result is

Theorem 1.1. *If Eq.* (1.1) *has a weak solution in* $\mathcal{PSH}_{\omega,k}(M)\cap C(M)$*, then this solution is in* $C^{2,\alpha}$.

Similar to [19], we will consider the corresponding parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t}=\log\dfrac{\omega_u^k\wedge\omega^{n-k}}{\omega^n}-\log f(x,u), \\ u\big|_{M\times\{0\}}=u_0, \end{cases} \tag{1.2}$$

where we suppose $u_0\in\mathcal{PSH}_{\omega,k}(M)\cap C^{2,\alpha}(M)$. Here $u_0\in\mathcal{PSH}_{\omega,k}(M)$ means that u_0 is an (ω,k)-plurisubharmonic function. The regularity of the weak solution follows from an approximation argument based on the regularity of the flow. The definition of (ω,k)-plurisubharmonic function can be found in Definition 2.1. We will prove the short time existence and regularity in Section 3, and then prove the main result in Section 4.

2 Preliminaries

In this section, we introduce some notions and some recent results of $(\omega,k)-$plurisubharmonic functions, following [2] (see also [8]).

First, we recall the elementary symmetric functions (see [2,5,10]). Let $\sigma_k(\lambda_1,\cdots,\lambda_n)$ be the k-th elementary symmetric function, i.e.

$$\sigma_k(\lambda_1,\cdots,\lambda_n)=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k},$$

where $(\lambda_1,\cdots,\lambda_n)\in\mathbb{R}^n$, and $1\le k\le n$. Let $\lambda_\omega\{a_{i\bar{j}}\}$ denote the eigenvalues of Hermitian symmetric matrix $\{a_{i\bar{j}}\}$ with respect to the Kähler form ω. We define

$$\sigma_k(a_{i\bar{j}})=\sigma_k\left(\lambda_\omega\{a_{i\bar{j}}\}\right).$$