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On the Positivity of Scattering Operators for Poincaré-Einstein Manifolds

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

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Abstract. In this paper, we mainly study the scattering operators for a Poincaré-Einstein manifold (X^{n+1},g_+), which define the fractional GJMS operators $P_{2\gamma}$ of order 2γ for $0<\gamma<\frac{n}{2}$ for the conformal infinity $(M,[g])$. We generalise Guillarmou-Qing's positivity results in [8] to the higher order case. Namely, if (X^{n+1},g_+) $(n\geq 5)$ is a hyperbolic Poincaré-Einstein manifold and there exists a smooth representative g for the conformal infinity such that the scalar curvature R_g is a positive constant and Q_4 is semi-positive on (M,g), then $P_{2\gamma}$ is positive for $\gamma\in[1,2]$ and the first real scattering pole is less than $\frac{n}{2}-2$.

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1 Introduction

Let $\overline{X}^{n+1}$ be a smooth compact manifold with boundary $\partial X=M$ and x be a smooth boundary defining function, i.e.

$$0\leq x\in C^\infty(\overline{X}),\quad M=\{x=0\},\quad dx|_M\neq 0.$$

We call (X^{n+1},g_+) a Poincaré-Einstein manifold with conformal infinity $(M,[g])$, if g_+ is a smooth Riemannian metric in the interior X which satisfies

$$\begin{cases} Ric_{g_+}=-ng_+ & \text{in } X,\\ x^2g_+|_{TM}\in[g] & \text{on } M.\end{cases}$$

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Here we require that x^2g_+ can be $C^{k,\alpha}$ extended to the boundary for some $k\geq 2$, $0<\alpha<1$. By the boundary regularity theorem given in [4], without loss of generality, we will assume $k=\infty$ for n odd and $k\geq n-1$ for n even in this paper. A straightforward calculation shows that all the sectional curvatures of (X^{n+1},g_+) converge to -1 when approaching to the boundary. A standard example is the hyperbolic space $\mathbb{H}^{n+1}$ in the ball model:

$$X^{n+1}=\{x\in\mathbb{R}^{n+1}:|z|<1\},\qquad g_+=\frac{4dz^2}{(1-|z|^2)^2}=\frac{4(dr^2+r^2d\theta^2)}{(1-r^2)^2},$$

where (r,θ) is the polar coordinates. Take the geodesic normal defining function $x=\frac{2(1-r)}{1+r}$. Then for $x\in(0,2)$

$$g_+=x^{-2}\left(dx^2+\left(1-\frac{x^2}{4}\right)^2d\theta^2\right),\qquad x^2g_+|_{T\mathbb{S}^n}=d\theta^2.$$

The spectrum and resolvent for the Laplacian-Beltrami operator of (X^{n+1},g_+) is studied by Mazzeo-Melrose [13], Mazzeo [14] and Guillarmou [7]. Actually the authors dealt with more general asymptotically hyperbolic manifolds. They showed that $\mathrm{Spec}(\triangle_+)=\sigma_{pp}(\triangle_+)\cup\sigma_{ac}(\triangle_+)$, where $\sigma_{pp}(\triangle_+)$ is the L^2-eigenvalue set and $\sigma_{ac}(\triangle_+)$ is the absolute spectrum, and

$$\sigma_{pp}(\triangle_+)\subset\left(0,\frac{n^2}{4}\right),\qquad \sigma_{ac}(\triangle_+)=\left[\frac{n^2}{4},+\infty\right).$$

For $s\in\mathbb{C},\mathrm{Re}(s)>\frac{n}{2},s(n-s)\notin\sigma_{pp}(\triangle_+)$, the resolvent $R(s)=(\triangle_+-s(n-s))^{-1}$ defines a bounded map $R(s):L^2(dV_{g_+})\longrightarrow L^2(dV_{g_+})$. Moreover $R(s)$ can be meromorphically extended to $\mathbb{C}\backslash\{\frac{n-1}{2}-N-\mathbb{N}_0\}$. Here N is an integer such that in the boundary asymptotical expansion of g_+, only even order terms appear up to order $2N$. For a Poincaré-Einstein metric g_+, if there is a smooth representative g on the conformal infinity, then according to the regularity result given in [4], $N\geq\frac{n-1}{2}$ for n odd and $N\geq\frac{n-2}{2}$ for n even. For hyperbolic space $\mathbb{H}^{n+1}$, $N=+\infty$. The L^2-eigenvalues can be estimated under certain geometric assumptions. For example in [11], Lee showed if (X^{n+1},g_+) is Poincaré-Einstein and its conformal infinity is of nonnegative Yamabe type, then $\sigma_{pp}(\triangle_+)=\emptyset$.

The scattering operators associated to (X^{n+1},g_+) are defined in the following way. Consider

$$(\triangle_+-s(n-s))u=0,\qquad x^{s-n}u|_M=f\in C^\infty(M).$$

If $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>\frac{n}{2}$, $s(n-s)\notin\sigma_{pp}(\triangle_+)$ and $2s-n\notin\mathbb{N}$, then

$$u=x^{n-s}F+x^sG,\quad F,G\in C^{k,\alpha}(\overline{X}),\quad F|_M=f.$$

We define the scattering operator $S(s)$ by

$$S(s):C^\infty(M)\longrightarrow C^\infty(M),\quad S(s)f=G|_M.$$