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Graham-Witten's Conformal Invariant for Closed Four Dimensional Submanifolds

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. It was proved by Graham and Witten in 1999 that conformal invariants of submanifolds can be obtained via volume renormalization of minimal surfaces in conformally compact Einstein manifolds. The conformal invariant of a submanifold Σ is contained in the volume expansion of the minimal surface which is asymptotic to Σ when the minimal surface approaches the conformaly infinity. In the paper we give the explicit expression of Graham-Witten's conformal invariant for closed four dimensional submanifolds and find critical points of the conformal invariant in the case of Euclidean ambient spaces.

AMS subject classifications: 53C42

Key words: Minimal surface, AdS/CFT, conformal invariant.

1 Introduction

In the introduction we give a description of the main result and some related background of the paper. The terminologies used in the introduction will be recalled in the next section.

Let (X^{d+1},g_+) be a conformally compact Einstein manifold and $(M^d,[g_{confinf}])$ its conformal infinity. A given metric $\overline{g}\in[g_{confinf}]$ uniquely determines a special defining function r on a neighborhood of M in $\overline{X}$, upon to the conditions that $(r^2g_+)|_M=\overline{g}$ and $|dr|_{r^2g_+}=1$ [12]. We denote $g_c=r^2g_+$. With the special defining function r, one can identify $M\times[0,\epsilon)$, for some $\epsilon>0$, with a neighborhood of M in $\overline{X}$. We denote the neighborhood by X_ϵ, and the identification

$$M\times[0,\epsilon)\cong X_\epsilon \tag{1.1}$$

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is defined as follows: $(p,r)\in M\times[0,\epsilon)$ corresponds to the point obtained by following the flow of $\nabla^{g_c}r$ emanating from p for r units of time. g_c on $M\times[0,\epsilon)$ takes the form of

$$g_c = dr^2+\widetilde{g}, \tag{1.2}$$

where $\widetilde{g}$ is a 1-parameter family of metrics on M with the parameter r. By solving the Einstein equation $Ric(g_+)=-dg_+$, for d odd the expansion of $\widetilde{g}$ is of the form

$$\widetilde{g}=g^{(0)}+g^{(2)}r^2+(\text{even powers})+g^{(d-1)}r^{d-1}+g^{(d)}r^d+\cdots, \tag{1.3}$$

where $g^{(j)}$ are tensors on M and the dots stand for terms vanishing to higher order. For j even and $0\le j\le d-1$, the tensor $g^{(j)}$ is locally formally determined by the boundary value $g^{(0)}=\overline{g}$, but $g^{(d)}$ is formally undetermined; for d even the expansion is

$$\widetilde{g}=g^{(0)}+g^{(2)}r^2+(\text{even powers})+hr^d\log r+g^{(d)}r^d+\cdots, \tag{1.4}$$

where $g^{(j)}$ and h are locally formally determined for j even and $0\le j\le d-2$ by $g^{(0)}=\overline{g}$.

The main object of the paper is a minimal surface in the conformally compact Einstein manifold (X,g_+) with prescribed asymptotic boundary. Let Σ^n be a submanifold of M and $Y^{n+1}\hookrightarrow(X,g_+)$ be a minimal surface which is asymptotic to Σ. The problem of existence and regularity of such minimal surfaces has been studied by Anderson [3, 4], Hardt-Lin [27], Lin [30–32], Tonegawa [35], Han-Jiang [25] and Han-Shen-Wang [26]. We denote

$$g=\overline{g}|_\Sigma. \tag{1.5}$$

The connections with respect to $(M,\overline{g})$ and (Σ,g) will be denoted by $\overline{\nabla}$ and ∇ respectively, and the connection of the normal bundle $T^\perp\Sigma$ of the immersion $\Sigma\hookrightarrow(M^d,\overline{g})$ will be denoted by $\nabla^\perp$.

Graham and Witten [16] have introduced a natural and useful way to reformulate Y. Namely, near the boundary M they express Y as a graph over $\Sigma\times[0,\epsilon)$ and expand the height functions of the graph in r. Near a point of Σ^n, let (x^i,y^α) be a local coordinate chart of M^d, where $1\le i\le n$ and $n+1\le\alpha\le d$, so that

$$\Sigma=\{y=0\};\quad \overline{g}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^\alpha})=0\ \text{ on } \Sigma\ ,\forall\ i,\alpha. \tag{1.6}$$

Note that via the identification (1.1), one has an extension of the coordinates (x^i,y^α) into X, which together with r forms a local coordinate chart of $\overline{X}$. The minimal surface Y can be written as a graph $\{y^\alpha=u^\alpha(x,r)\}$. That is, near the boundary $Y=(x^i,u^\alpha(x,r),r)$.

Graham and Witten [16] proved that for n odd

$$u=u^{(2)}r^2+(\text{even powers})+u^{(n+1)}r^{n+1}+u^{(n+2)}r^{n+2}+\cdots, \tag{1.7}$$

and for n even

$$u=u^{(2)}r^2+(\text{even powers})+u^{(n)}r^n+w_nr^{n+2}\log r+u^{(n+2)}r^{n+2}+\cdots, \tag{1.8}$$