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On 2-Microlocal Herz Type Besov and Triebel-Lizorkin Spaces with Variable Exponents

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Abstract. In this paper, 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents are introduced for the first time. Then, we give characterizations of these spaces by so-called Peetre's maximal functions. Further, the atomic and molecular decompositions of these spaces are obtained. Finally, using the characterizations of the spaces by local means and molecular decomposition we obtain the wavelet characterizations.

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Key words: 2-microlocal, variable exponent, Herz type spaces, Besov spaces, Triebel-Lizorkin spaces, Peetre's maximal operator.

1 Introduction

The theory of function spaces with variable exponents has developed rapidly in recent years (see, e.g., [2–3, 5–6, 8–9, 16–17, 19–21, 23–28]). It is worth noting that variable exponent Lebesgue spaces first appeared in [16] by Orlicz in 1931. In 2009, Izuki [7] defined Herz spaces with variable exponent $K^{\alpha,q}_{p(\cdot)}$ and obtained wavelet characterization of those spaces by virtue of the result on $L^{p(\cdot)}(\mathbb{R}^n)$ [6, 13]. Moreover, Izuki [8-10] proved the boundedness of some sublinear operators and commutators on $K^{\alpha,q}_{p(\cdot)}$.

As a continuation of the work for Herz spaces with variable exponent, Shi and Xu [21] introduced Herz type Besov and Triebel-Lizorkin spaces with variable exponent, $K^{\alpha,q}_{p(\cdot)}B^s_\beta$ and $K^{\alpha,q}_{p(\cdot)}F^s_\beta$, and obtained their equivalent quasi-norms. Subsequently, Dong and Xu gave characterizations of $K^{\alpha(\cdot),q}_{p(\cdot)}B^s_\beta$ and $K^{\alpha(\cdot),q}_{p(\cdot)}F^s_\beta$ by Peetre's maximal functions in [4].

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On the other hand, the concept of 2-microlocal spaces has aroused the interest of some scholars, and this concept initially appeared in the book of Peetre [18]. 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability were studied first by Kempka in [11–12]. With even q variable in the Besov case, Almeida and Caetano [1–2] study various key properties for 2-microlocal Besov and Triebel-Lizorkin spaces with all exponents variable, including Sobolev type embeddings, atomic and molecular representations.

In this paper, combining its of Herz type spaces and 2-microlocal spaces, the authors introduce 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents (see Section 2), which is the extension of 2-microlocal Besov and Triebel-Lizorkin spaces and Herz type Besov and Triebel-Lizorkin spaces with variable exponents. In Section 3, we give a simple proof for the characterizations of these spaces by Peetre's maximal functions and the local means. Another of the main results demonstrates an embedding theorem about two sequence spaces and clarify some convergence issues. Then applying the convergence and the characterization of Peetre's maximal functions, it is asserted that the atomic, molecular and wavelet characterizations of these spaces in Sections 4 and 5.

2 Preliminaries and definitions

In this section, we introduce the basic notation in the theory of 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents.

Definition 2.1. Let $p:\mathbb{R}^n \to [1,\infty)$ be a measurable function. $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty \right\}$$

and

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Denote $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions p on $\mathbb{R}^n$ with range in $[1,\infty)$ such that

$$1 < p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p^+ < \infty.$$

Moreover, we define $\mathcal{P}^0(\mathbb{R}^n)$ to be the set of all measurable functions p on $\mathbb{R}^n$ with range in $(0,\infty)$ such that

$$0 < p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p^+ < \infty.$$