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A Note on Gaussian BV Function and Its Heat Semigroup Characterization

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Abstract. In this note, we investigate the properties of Gaussian BV functions and give a heat semigroup characterization of BV functions in Gauss space. In particular, the latter is the nontrivial generalization of classical De Giorgi's heat kernel characterization of function of bounded variation on Euclidean space to the case of Gauss space.

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1 Introduction

There are various definitions of variational functions, and the related class of bounded variational functions (or BV functions for short), is meaningful in different contexts and equivalent in general. On the Euclidean space, the variation of $f \in L^1(\mathbb{R}^n)$ with the Lebesgue measure can be defined as

$$\|Df\|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}\varphi dx : \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \quad \text{with} \quad \|\varphi\|_{L^\infty} \leq 1 \right\},$$

where $\operatorname{div}\varphi(x) := \sum_{i=1}^n \frac{\partial \varphi_i}{\partial x_i}$. In fact, the original definition of the variation of a function was given by De Giorgi through a thermonuclear regularization process (see [3, 4]). He also proved that

$$\|Df\|(\mathbb{R}^n) = \lim_{t \to 0} \int_{\mathbb{R}^n} |\nabla T_t f| dx, \tag{1.1}$$

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where ∇ denotes the gradient of the function f, and

$$T_t f(x) = \int_{\mathbb{R}^n} h(t,x-y) f(y) dy$$

is the heat semigroup with the Gauss-Weierstrass kernel $h(t,x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$. Later, Miranda, Pallara, Paronetto and Preunkert in [6] proved the equality (1.1) on the Riemannian manifold M

$$|Du|(M) = \lim_{t\to 0} \int_M |dP_t u| dV$$

with two geometric assumptions: the Ricci curvature is bounded from below and the volume of geodesic balls of fixed radius has a positive lower bound which does not depend on the center, where $\{P_t\}_{t\geq 0}$ is the heat semigroup generated by the Laplace-Beltrami operator on M. After that, Carbonaro and Mauceri proved the equality (1.1) based on properties of heat semigroups with the only restriction that the Ricci curvature is bounded from below in [1]. [2] implies that the equality (1.1) holds in a weaker sense and the authors provide two different characterizations of sets with finite perimeter and functions of bounded variation in Carnot groups.

In order to state our main result, we recall some basic facts for the n dimensional Gauss space $\mathbb{G}^n$. This space is equipped with the following measure

$$\gamma(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}, \quad \forall x \in \mathbb{R}^n,$$

the Gaussian volume element $dV_\gamma = \gamma dx$ and the γ-divergence $\mathrm{div}_\gamma \varphi = \mathrm{div}\varphi - x\cdot\varphi$, $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$. Next we also recall the Gaussian BV functions and its properties. For any open subset $\Omega \subseteq \mathbb{R}^n$, the γ-total variation of $f \in L^1(\Omega)$ is defined by

$$\|Df\|(\Omega;\mathbb{G}^n) = \sup\left\{ \int_\Omega f \mathrm{div}_\gamma \varphi dV_\gamma : \varphi \in C_c^1(\Omega,\mathbb{R}^n) \ \text{ with } \ \|\varphi\|_{L^\infty} \leq 1 \right\},$$

where $\|\varphi\|_{L^\infty} = \mathrm{esssup}_{x\in\Omega} (|\varphi_1|^2 + \cdots + |\varphi_n|^2)^{1/2}$. Particularly, if $\Omega = \mathbb{R}^n$, we denote $\|Df\|(\Omega;\mathbb{G}^n)$ by $\|Df\|(\mathbb{G}^n)$. The function $f \in L^1(\Omega)$ is of the Gaussian bounded variation on Ω and denoted by $f \in BV(\Omega;\mathbb{G}^n)$ if

$$\|Df\|(\Omega) < \infty.$$

When $\Omega = \mathbb{R}^n$, we denote $BV(\Omega;\mathbb{G}^n)$ by $BV(\mathbb{G}^n)$. The space $BV_{\mathrm{loc}}(\mathbb{G}^n)$ is said to be of locally Gaussian bounded variation in $\mathbb{R}^n$, if

$$\|Df\|(N;\mathbb{G}^n) < \infty,$$

for every open set $N \subseteq \mathbb{R}^n$ and $\overline{N}$ is compact. For a set $E \subseteq \mathbb{R}^n$, $P_\gamma(E) := \|D\chi_E\|(\mathbb{G}^n)$ be the Gaussian perimeter of E. Refer to [5] for some properties of $P_\gamma(E)$. In particular, from [5] we know that the Gauss-Green formula is valid:

$$\int_E \mathrm{div}_\gamma v dV_\gamma = \int_{\partial^* E} v \cdot \nu_E dP_\gamma, \quad \forall v \in C_c^1(\mathbb{R}^n,\mathbb{R}^n),$$