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Non-Negative Integer Matrix Representations of a $\mathbb{Z}_+$-ring

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Abstract. The $\mathbb{Z}_+$-ring is an important invariant in the theory of tensor category. In this paper, by using matrix method, we describe all irreducible $\mathbb{Z}_+$-modules over a $\mathbb{Z}_+$-ring $\mathcal{A}$, where $\mathcal{A}$ is a commutative ring with a $\mathbb{Z}_+$-basis $\{1, x, y, xy\}$ and relations:

$$x^2=1, \ \ y^2=1+x+xy.$$

We prove that when the rank of $\mathbb{Z}_+$-module $n \geq 5$, there does not exist irreducible $\mathbb{Z}_+$-modules and when the rank $n \leq 4$, there exists finite inequivalent irreducible $\mathbb{Z}_+$-modules, the number of which is respectively 1, 3, 3, 2 when the rank runs from 1 to 4.

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Key words: Non-negative integer, matrix representation, irreducible $\mathbb{Z}_+$-module, $\mathbb{Z}_+$-ring.

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1 Introduction

Tensor categories are usually thought as counterparts of groups and rings in the world of categories. They are ubiquitous in noncommutative algebra and representation theory. The $\mathbb{Z}_+$-ring is an important invariant in the theory of tensor category. The terminology on $\mathbb{Z}_+$-rings comes from the paper by Lusztig [1] as well as [2, 3]. Such rings were also studied by Davydov in [4, 5]. The basic concepts and facts about $\mathbb{Z}_+$-rings can be found in [3, 6]. Examples of $\mathbb{Z}_+$-rings include the Green rings of Hopf algebras [7, 8, 9, 10, 11, 12] and the Grothendieck rings of tensor categories [13, 14, 15, 16, 17]. The terminology on $\mathbb{Z}_+$-modules (or $\mathbb{Z}_+$-representations) is from [2, 3, 6] which is different from the

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common representations over rings or algebras. A $\mathbb{Z}_+$-module M is called irreducible if any $\mathbb{Z}_+$-submodule of M is 0 or M. A $\mathbb{Z}_+$-module M is called indecomposable if it is not equivalent to a direct sum of two nonzero $\mathbb{Z}_+$-modules. A $\mathbb{Z}_+$-module over a based ring satisfying the rigid condition is called a based module in [3] or a NIM-representation (the non-negative integer matrix representation) in [18, 19, 20, 21]. It is quite difficult to classify all irreducible $\mathbb{Z}_+$-modules but the job is meaningful and interesting.

In [3], Ostrik proved that for a given $\mathbb{Z}_+$-ring of finite rank there exist only finite inequivalent irreducible $\mathbb{Z}_+$-modules. In the proofs, Ostrik indicated the rank of any irreducible $\mathbb{Z}_+$-module has an upper bound. However, for many situations, the upper bound seems quite large. For a given $\mathbb{Z}_+$-ring, it is difficult to determine whether the rank of an irreducible $\mathbb{Z}_+$-module is less than or equal to the rank of this ring. In this paper, by using matrix method, we describe all irreducible inequivalent $\mathbb{Z}_+$-modules over a $\mathbb{Z}_+$-ring $\mathcal{A}$, where $\mathcal{A}$ is a commutative $\mathbb{Z}_+$-ring with a $\mathbb{Z}_+$-basis $\{1,\ x,\ y,\ xy\}$ and relations: $x^2=1$, $y^2=1+x+xy$. In addition, we find the rank of each irreducible $\mathbb{Z}_+$-module is less than or equal to the rank of $\mathcal{A}$. In fact, the problem is equivalent to studying the irreducible NIM solutions to a system of matrix equations:

$$\begin{cases} A^2=E, \\ B^2-A-E-AB=0, \\ AB=BA. \end{cases} \tag{$*$}$$

We analyze all the situations when the order of the matrix is n, and we get the results completely. When the rank of $\mathbb{Z}_+$-modules $n\geq 5$, there does not exist irreducible $\mathbb{Z}_+$-modules and when the rank $n\leq 4$, there exists finite inequivalent irreducible $\mathbb{Z}_+$-modules, the number of which is respectively 1, 3, 3, 2 when the rank runs from 1 to 4.

The rest of the paper is outlined as follows. In Section 2, we recall some relevant theorems and prove the preliminary propositions we will use in the next sections. In Section 3, we first exhibit the system of matrix equations, then we prove that there does not exist irreducible NIM solutions when order $n\geq 5$. In Section 4, we exhibit the concrete irreducible NIM solutions when order $n\leq 4$. More explicitly, the irreducible $\mathbb{Z}_+$-module over this $\mathbb{Z}_+$-ring $\mathcal{A}$ exists if and only if the rank of module is less than or equal to 4. The concrete irreducible NIM representations can be seen in table 4.1. Furthermore, in each case, we classify the irreducible $\mathbb{Z}_+$-modules under the equivalence in Table 4.2.

2 Preliminaries

2.1 Basic definitions and notation

2.1.1 The theory of matrix

We assume all matrices in this paper belong to $\mathbb{M}_n(\mathbb{N})$, where $\mathbb{M}_n(\mathbb{N})$ means the set consisting of n-order square matrices with only natural number elements. For any matrix A, we denote the element at the i-th row and j-th column of A by a_{ij}. If $a_{ij}>0$ for all i,