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# Nontrivial Solution for a Kirchhoff Type Problem with Zero Mass

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**Abstract.** Consider the Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\left(\frac{1}{|x|^\mu}*F(u)\right)f(u)\ \text{ in }\mathbb{R}^N,\ \ u\in D^{1,2}(\mathbb{R}^N), \tag{0.1}$$

where $a>0$, $b\geq 0$, $0<\mu<\min\{N,4\}$ with $N\geq 3$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and $F(u)=\int_0^u f(t)dt$. Under some general assumptions on $f$, we establish the existence of a nontrivial spherically symmetric solution for problem (0.1). The proof is mainly based on mountain pass approach and a scaling technique introduced by Jeanjean.

**AMS subject classifications**: 35A15, 35J60.

**Key words**: Kirchhoff type equation, zero mass, mountain pass approach.

## 1 Introduction

This paper is concerned with the Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\left(\frac{1}{|x|^\mu}*F(u)\right)f(u)\ \text{ in }\mathbb{R}^N,\ \ u\in D^{1,2}(\mathbb{R}^N), \tag{1.1}$$

where $a>0$, $b\geq 0$, $0<\mu<\min\{N,4\}$ with $N\geq 3$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and $F(u)=\int_0^u f(t)dt$. Equations of this type are derived as models of several physical phenomena (see, for example, [8, 14, 18, 20, 21]) and have been studied extensively in recent years.

The so called "zero mass" case (that is, roughly speaking, when $f'(0)=0$) is of particular interest in the current paper. In such a case, (1.1) is closely related to Yang-Mills equations. In the context of semilinear elliptic equation

$$-\Delta u=g(u)\ \text{ in }\mathbb{R}^N, \tag{1.2}$$

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a large amount of papers are devoted to the existence, multiplicity and properties of nontrivial solutions. However, we are only aware of a few papers related to "zero mass" case. Some existence results of problem (1.2) were proved in [3,23] for the particular case where $g(u)=|u|^{\frac{4}{N-2}}u$ and in [1, 5–8] for the general case where $g$ is supercritical at the origin and subcritical at infinity.

Kirchhoff type problem has also attracted much attention in recent years. Many existence and multiplicity results have been established by classical variational methods (see, for example, [9,11,15,17,22]). However, "zero mass" case is relatively less studied. In [4], Azzollini considered the Kirchhoff type equation

$$-M\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=g(u)\ \text{ in }\mathbb{R}^N \tag{1.3}$$

and proved that (1.3) has a positive solution if $g$ is a (possibly "zero mass") Berestycki-Lions nonlinearity (see [4, Definition 0.2] or [8]). In [16], Li, Li and Shi studied (1.3) with $M(t)=a+bt$ and $g(u)$ replaced by $K(x)g(u)$. They established two existence results for the problem if $b\geq 0$ is suitably small and $g$ is supercritical at the origin and superlinear and subcritical at infinity.

In [2], Alves and Yang considered (1.1) with $b=0$ and $N=3$. They proved the existence of a nontrivial solution if $f$ is a "zero mass" Berestycki-Lions type nonlinearity (in fact, they made an additional assumption on the nonlinearity: $f(t)t-F(t)\geq 0$ for all $t\in\mathbb{R}$). We also would like to refer interested readers to [10] for (1.1) with $b=0$ and $f$ being the critical pure power nonlinearity. Motivated by above works, we consider (1.1) and assume that $f\in C(\mathbb{R},\mathbb{R})$ satisfies

$(f_1)$ $\lim_{t\to 0}f(t)/|t|^{\frac{N+2-\mu}{N-2}}=\lim_{t\to\infty}f(t)/|t|^{\frac{N+2-\mu}{N-2}}=0$;

$(f_2)$ there exists $\xi\in\mathbb{R}\setminus\{0\}$ such that $F(\xi):=\int_0^{\xi}f(t)dt\neq 0$.

Our main result is as follows.

**Theorem 1.1.** Let $a>0$, $b\geq 0$ and $0<\mu<\min\{N,4\}$ with $N\geq 3$. If $f$ satisfies $(f_1)$ and $(f_2)$, then problem (1.1) has at least a nontrivial spherically symmetric solution.

**Remark 1.2.** Theorem 1.1 is a complement of the results in [4, 16], where only local nonlinearity is taken into account. Moreover, we do not assume that $b\geq 0$ is suitably small and the nonlinearity is superlinear at infinity as in [16]. Theorem 1.1 is also a generalization of [2, Theorem 1.1] even for the particular case $b=0$, because we do not assume that $f$ satisfies $f(t)t-F(t)\geq 0$ for all $t\in\mathbb{R}$.

Throughout this paper, we use $\|\cdot\|$ and $|\cdot|_p$ to denote the usual norms in $D^{1,2}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ with $p\geq 1$, respectively. We also set $D_r^{1,2}(\mathbb{R}^N)=\{u\in D^{1,2}(\mathbb{R}^N)\,|\,u(x)=u(|x|)\}$ and $B_R=\{x\in\mathbb{R}^N\,|\,|x|\leq R\}$. The symbol $o_n(1)$ means a quantity tending to 0 as $n\to\infty$. The letters $C$, $C_j$ and $C_\varepsilon$ stand for positive constants which may take different values at different places.