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Boundedness of Bilinear Fractional Integral Operators on Vanishing Generalized Morrey Spaces

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Abstract. In this paper, we establish the boundedness of the bilinear fractional integral operator B_α and the subbilinear fractional maximal operator M_α on vanishing generalized Morrey spaces $V_0L^{p,\varphi}(\mathbb{R}^n)$, $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ and $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$. The main novelty of this article is that we control B_α by the subbilinear maximal operator M and $M_{\alpha'}$ with $\alpha' > \alpha$. Some specific examples for the main results of this paper are also included.

AMS subject classifications: 26A33, 42B35, 42B25, 42B20

Key words: Bilinear fractional integral operator, subbilinear fractional maximal operator, generalized Morrey space, vanishing property.

1 Introduction

The theory of Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ was originated from the work of Morrey [24] and then systematically investigated by Campanato [8] and Peetre [25]. Since then, it is extensively applied to harmonic analysis and partial differential equations. As the limitation of our knowledge, we only list a few references here. We refer the reader to articles [3,4,9,20,26,29] and monographs [1,33] for the applications of the theory of Morrey spaces to harmonic analysis. For the applications of the theory of Morrey spaces to potential analysis and partial differential equations, please see papers [2,17,19,20,28,30] and monographs [18,31,32]. The theory of Morrey spaces was also applied to the interpolation theory, see, for instance, [12,13,21,23]. The boundedness of some classical bilinear operators on Morrey spaces can be found in [10,14–16]. For the recent development of the theory of vanishing (generalized) Morrey spaces, we refer the reader to [5–7,27].

We now introduce the notion of the classical Morrey spaces essentially contained in [24]. In what follows, for any $p\in(0,\infty)$, let $L^p_{\rm loc}(\mathbb{R}^n)$ denote *the space of all locally p-integrable functions on* $\mathbb{R}^n$. For any $p\in(0,\infty)$, $\lambda\in[0,n)$, the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$

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is defined to be the set of all functions f with their norms in $L^{p,\lambda}(\mathbb{R}^n)$ defined by setting $||f||_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\frac{\lambda}{p}} ||f||_{L^p(B(x,r))} < \infty$, where, for any $x\in\mathbb{R}^n$ and $r\in(0,\infty)$, $B(x,r) := \{y\in\mathbb{R}^n : |y-x|<r\}$. It is well known that, if $\lambda=0$, then $L^{p,\lambda}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ for any given $p\in(0,\infty)$.

We now recall the notion of the generalized Morrey spaces from [27] (see the references in [27] for more information). The *generalized Morrey space* $L^{p,\varphi}(\mathbb{R}^n)$, where $p\in(0,\infty)$ and φ: $\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ is a measurable function, is defined to be the set of all functions $f\in L^p_{\rm loc}(\mathbb{R}^n)$ with their norms in $L^{p,\varphi}(\mathbb{R}^n)$ defined by setting

$$||f||_{L^{p,\varphi}(\mathbb{R}^n)} := \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left[\frac{1}{\varphi(x,r)}\int_{B(x,r)} |f(y)|^p dy\right]^{\frac{1}{p}} < \infty.$$

In this paper, we consider the following subspaces of $L^{p,\varphi}(\mathbb{R}^n)$ for any $p\in(0,\infty)$ and measurable function φ: $\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. The *space* $V_0L^{p,\varphi}(\mathbb{R}^n)$ introduced by Samko [27] is defined to be the set of all functions $f\in L^{p,\varphi}(\mathbb{R}^n)$ satisfying the *vanishing property at zero* (for short, the V_0 *property*) : $\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} m_{p,\varphi}(f;x,r)=0$, where, for any $x\in\mathbb{R}^n$ and $r\in(0,\infty)$, $m_{p,\varphi}(f;x,r) := [1/\varphi(x,r)]\int_{B(x,r)}|f(y)|^p dy$.

As in [27, (3.3) and (3.4)], to guarantee that all bounded functions with compact support belong to $V_0L^{p,\varphi}(\mathbb{R}^n)$, we impose the following conditions on the function φ in the definition of $V_0L^{p,\varphi}(\mathbb{R}^n)$:

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{r^n}{\varphi(x,r)}=0 \tag{1.1}$$

and

$$\inf_{r\in(1,\infty)}\sup_{x\in\mathbb{R}^n}\varphi(x,r)>0. \tag{1.2}$$

The (quasi-)norm of f in $V_0L^{p,\varphi}(\mathbb{R}^n)$ is defined by setting

$$||f||_{L^{p,\varphi}(\mathbb{R}^n)} := \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left[m_{p,\varphi}(f;x,r)\right]^{1/p}.$$

Another two subspaces of $L^{p,\varphi}(\mathbb{R}^n)$ are generalized from the spaces $V_\infty L^{p,\lambda}(\mathbb{R}^n)$ and $V^{(*)}L^{p,\lambda}(\mathbb{R}^n)$ in [5]. The *space* $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ is defined to be the set of all functions $f\in L^{p,\varphi}(\mathbb{R}^n)$ having the vanishing property at infinity (for short, the V_∞ *property*), namely,

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n} m_{p,\varphi}(f;x,r)=0. \tag{1.3}$$

To guarantee that all bounded functions with compact support belong to $V_\infty L^{p,\varphi}(\mathbb{R}^n)$, we assume that the function φ in the definition of $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ satisfies (1.2) and

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi(x,r)}=0. \tag{1.4}$$