J. Math. Study
doi: 10.4208/jms.v55n2.22.03

Vol. **55**, No. 2, pp. 139-157
June 2022

Modeling and Applications to Circular Data with a Wrapped Poisson-Lindley Model

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Received April 12, 2021; Accepted July 7, 2021;
Published online February 10, 2022.

Abstract. In the recent years, researchers in many fields, including meteorology, economics, sociology, psychology, and epidemiology, have shown a keen interest in the analysis and modeling of wrapped data. This motivates the development of new wrapped distributions and related models. In this paper, a simple and new wrapped model based on the Poisson-Lindley distribution is developed. Many of its properties are obtained, such as probability mass and cumulative distribution functions, survival and hazard rate functions, and probability generating function. The estimation of the model parameter is investigated by the maximum likelihood method. Test and evaluation statistics are also considered to assess the performance of the distribution among the most frequently wrapped discrete probability models using three different circular practical data sets.

AMS subject classifications: 60E05, 62E15, 62F10

Key words: Wrapped distributions, Poisson-Lindley distribution, estimation, circular data, evaluating tests.

1 Introduction

In the current era, many diverse statistical fields are emerging, such as biostatistics, environmental statistics, geostatistics, statistical mechanics, statistical computing and management statistics. However, very little attention is paid to the directional statistics that deal with direction axes or rotations. Such directional statistics include the direction of the earth's magnetic pole, the direction of flight of a bird or the orientation of an animal,

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the arrival times of patients in an emergency clinic, incidences of a disease throughout the year, and the number of tourists (daily or monthly) in a city within a year, where the calendar is regarded as a one-year clock, the daily wind directions, the ocean current directions, departure directions of animals, direction of bone-fracture planes, and orientation of bees in a beehive after stimuli, etc.

The directional statistics mentioned above have many innovative landscapes, both in terms of modeling and statistical treatment. For this purpose, researchers have to adopt different ways to obtain circular distributions. One of them is wrapping a linear distribution around the unit circle. In this regard, [11] is the pioneer of wrapped distributions. Since this innovation, a lot of work has been done in this field. Some of them include the wrapped normal (WN) distribution, wrapped Cauchy (WC) distribution, wrapped Poisson distribution by [7], wrapped exponential and Laplace distributions by [6], wrapped lognormal, logistic, Weibull, and extreme-value distributions by [14], wrapped skew Laplace distribution by [8], wrapped geometric distribution, wrapped gamma distribution by [13], wrapped variance gamma distribution by [1], wrapped binomial distribution by [4], wrapped log Kumaraswamy distribution by [9], discrete wrapped exponential and negative binomial distributions by [17, 18] and, recently, [10] introduced the wrapped Lindley distribution. As a matter of fact, the existing wrapped distributions literature lacks the discrete wrapped distributions. Moreover, probability functions of the presented wrapped models are also not in the closed form and, ultimately, mathematically, are not amenable.

Now, let us briefly recall the methodology for the construction of a new wrapped distribution. Let X be a discrete random variable with integer values. Then, the corresponding wrapped distribution is the distribution of the following random variable: $X_w = 2\pi X \ (mod 2\pi m)$, which has the support $\{2\pi r/m,\ r=0,1,\ldots,m-1\}$. We now suppose that X follows the Poisson-Lindley distribution with parameter $\theta>0$ introduced by [15] and characterized by the following probability mass function (pmf):

$$f(x;\theta)=\frac{\theta^2(x+\theta+2)}{(1+\theta)^{x+3}}, \quad x=0,1,2,\cdots. \tag{1.1}$$

It is of particular interest because of the following notable properties: tractable probability functions, unimodality, overdispersion, infinite divisibility and increasing hazard rate function (hrf). Also, in some circumstances, it provides a suitable alternative to the Poisson, geometric and negative binomial distributions (in particular, it is known to have smaller skewness and kurtosis than the negative binomial distribution). Also, the model in Eq. (1.1) was derived based on an interesting property of the linear exponential family of single parameter distributions (see [15]). It contains some interesting properties, including approximation to the negative binomial and Hermite distributions, and has a wide range of application in many fields, such as medicine, engineering, ecology and genetics (see [16]). Further, this distribution can be represented as a mixture of geometric with parameter $1/(1+\theta)$ and negative binomial with parameters 2 and $1/(1+\theta)$, where the mixing proportions are $\theta/(1+\theta)$ and $1/(1+\theta)$, respectively. More details on