J. Math. Study
doi: 10.4208/jms.v55n2.22.04

Vol. **55**, No. 2, pp. 158-179
June 2022

On Doubly Twisted Product of Complex Finsler Manifolds

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Received February 3, 2021; Accepted June 15, 2021;
Published online February 7, 2022.

Abstract. Let (M_1,F_1) and (M_2,F_2) be two strongly pseudoconvex complex Finsler manifolds. The doubly twisted product (abbreviated as DTP) complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is the product manifold $M_1\times M_2$ endowed with the twisted product complex Finsler metric $F^2=\lambda_1^2F_1^2+\lambda_2^2F_2^2$, where λ_1 and λ_2 are positive smooth functions on $M_1\times M_2$. In this paper, the relationships between the geometric objects (e.g. complex Finsler connections, holomorphic and Ricci scalar curvatures, and real geodesic) of a DTP-complex Finsler manifold and its components are derived. The necessary and sufficient conditions under which the DTP-complex Finsler manifold is a Kähler Finsler (respctively weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg) manifold are obtained. By means of these, we provide a possible way to construct a weakly complex Berwald manifold, and then give a characterization for a complex Landsberg metric that is not a Berwald metric.

AMS subject classifications: 53C60, 53C40

Key words: Doubly twisted products, complex Finsler metric, holomorphic curvature, geodesic

1 Introduction

Warped product and twisted product are important methods used to produce new class of geometrical spaces, and these products are widely applied in theoretical physics. The notion of warped product of two Riemannian manifolds was first introduced by O'Neill and Bishop to construct Riemannian manifolds with negative curvature [11], then it was studied by many authors [10,15,16,23]. Asanov considered the warped product of Finsler

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manifolds and obtained some models of relativity theory [3, 4]. Kozma, Peter and Varga extended the notion of warped product to the real Finsler manifolds [19]. Many properties of such kinds of products were studied (see [9, 19, 24, 35, 38]). In 2016, He and Zhong systematically studied the doubly warped product of complex Finsler manifolds [14], they got a new method of constructing weakly complex Berwald metric through the doubly warped product of complex Finsler metric.

The notion of twisted product of Riemannian manifolds was mentioned first by Chen in [13], and was generalized for the pseudo-Riemannian case by Ponge and Reckziegel [26]. Kozma, Peter, and Shimada extended the construction of twisted product to real Finslerian case [18], they presented the construction of twisted product of Finsler manifolds and investigate some geometrical properties relating to Cartan connection, geodesics and completeness. In [25], Peyghan, Tayebi and Nourmohammadi obtained the Riemannian curvature and some of non-Riemannian curvatures of the twisted product Finsler manifold such as Berwald curvature, mean Berwald curvature, and studied locally dually flat twisted product Finsler manifold. In [34], Wang gave the necessary and sufficient conditions for multiply twisted product Finsler manifolds to be Riemannian, Landsberg, Berwald, locally dually flat and locally Minkowski.

Recently, Zhong proved that there are lots of strongly pseudoconvex (even strongly convex) unitary invariant complex Finsler metrics in domains in $\mathbb{C}^n$ [37]. The purpose of this paper is to study the doubly twisted product of strongly pseudoconvex complex Finsler manifolds.

This paper is organized as follows. In Section 2, we recall some basic concepts and notations of complex Finsler geometry and extend the doubly twisted product to complex Finsler manifold. In Section 3, we deduce the most commonly used complex Finsler connections (the Chern-Finsler connection, the complex Rund connection, the complex Berwald connection, and the complex Hashiguchi connection, etc.) of the DTP-complex Finsler manifold, which are expressed by the corresponding connections of its components. In Section 4, we derive the formulae of the holomorphic curvature and Ricci scalar curvature of the DTP-complex Finsler manifold, which are expressed by the holomorphic and Ricci scalar curvatures of its components. In Section 5, we derive the real geodesic equations of the DTP-complex Finsler manifold. In Section 6, we obtain the necessary and sufficient conditions under which the DTP-complex Finsler manifold to be Kähler Finsler (respctively weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg, complex locally Minkowski) manifold. By using of these results, we provide a possible way to construct special complex Finsler manifolds.

2 Preliminary

In this section, we recall some basic concepts and notations which will be used in this paper, and give the definition of the DTP-complex Finsler manifolds.

Let M be a complex manifold of complex dimension n, and $T^{1,0}M$ be the holomorphic