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Highly Accurate Latouche-Ramaswami Logarithmic Reduction Algorithm for Quasi-Birth-and-Death Process

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Abstract. This paper is concerned with the quadratic matrix equation $A_0+A_1X+A_2X^2=X$ where $I-A_0-A_1-A_2$ is a regular M-matrix, i.e., there exists an entrywise positive vector $\boldsymbol{u}$ such that $(I-A_0-A_1-A_2)\boldsymbol{u}\geq 0$ entrywise. It broadly includes those originally arising from the quasi-birth-and-death (QBD) process as a special case where $I-A_0-A_1-A_2$ is an irreducible singular M-matrix and $(A_0+A_1+A_2)\mathbf{1}=\mathbf{1}$ with $\mathbf{1}$ being the vector of all ones. A highly accurate implementation of Latouche-Ramaswami logarithmic reduction algorithm [*Journal of Applied Probability*, 30(3):650–674, 1993] is proposed to compute the unique minimal nonnegative solution of the matrix equation with high entrywise relative accuracy as it deserves. Numerical examples are presented to demonstrate the effectiveness of the proposed implementation.

AMS subject classifications: 15A24, 65F30, 65H10

Key words: Quadratic matrix equation, M-matrix, quasi-birth-and-death process, minimal nonnegative solution, entrywise relative accuracy.

1 Introduction

In the quasi-birth-and-death (QBD) process, the following quadratic matrix equation [17, 25]

$$A_0+A_1X+A_2X^2=X, \tag{1.1}$$

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plays a vital role in analyzing the process, where A_i for $i=0,1,2$ are $n\times n$ nonnegative matrices, sitting as blocks in an infinite block-tridiagonal transition matrix. In the QBD process, $I-A_0-A_1-A_2$ is irreducible and singular, and, furthermore,

$$(A_0+A_1+A_2)\mathbf{1}_n=\mathbf{1}_n, \tag{1.2}$$

where $\mathbf{1}_n$ (often simply $\mathbf{1}$ when its dimension is clear from the context) is the column vector of all ones of dimension n. Under these conditions, Eq. (1.1) admits a unique minimal nonnegative solution [23] (see also [6]), denoted by Φ hereafter, in the sense that

$$\Phi\le X \quad \text{for any other nonnegative solution } X \text{ of equation (1.1).}$$

Existing numerical methods for computing the solution Φ include fixed point iterations (e.g., [8, 10] and references therein), Newton's method, the Latouche-Ramaswami logarithmic reduction algorithm (called the LR algorithm hereafter), the method of cyclic reduction, and doubling algorithms (see, e.g., [5, 6, 11, 13, 15, 16, 20, 30] and references therein). The fixed point iterations are usually linearly convergent and can suffer very slow convergence when $\rho(\Phi)$ is almost 1, or no convergence when it is 1. Newton's method needs to solve a Sylvester equation in each of its iterative steps, which, unfortunately, can be as expensive as solving equation (1.1) itself by other methods and thus is not competitive. It has been observed that the method of cyclic reduction is equivalent to the LR algorithm [11,16] which turns out to be a very efficient method nowadays. The application of doubling algorithms [14] to solve equation (1.1) from the QBD process and beyond is more recent [6] and their efficiency is about the same as the LR algorithm.

Because of (1.2) and that $A_0+A_1+A_2$ is nonnegative and irreducible, there exists a positive vector $\mathbf{z}$ [22, p.673] such that $\mathbf{z}^{\mathrm{T}}(A_0+A_1+A_2)=\mathbf{z}^{\mathrm{T}}$. The associated QBD process is further classified into three categories [17]: positive recurrent if $\mathbf{z}^{\mathrm{T}}(A_0-A_2)\mathbf{1}>0$, null recurrent if $\mathbf{z}^{\mathrm{T}}(A_0-A_2)\mathbf{1}=0$, and transient if $\mathbf{z}^{\mathrm{T}}(A_0-A_2)\mathbf{1}<0$. Except for the fixed point iterations, all other methods mentioned in the previous paragraph are quadratically convergent unless the QBD process is null recurrent [11,30].

Before the work of Ye [30], it was noted that computed Φ by the LR algorithm can suffer heavy accuracy loss, especially when the QBD process is nearly null recurrent. It turns out that inaccurate numerical matrix inversions during the LR iterative process are to blame because the involved matrices are increasingly singular and thus increasingly ill-conditioned for inversions as the iteration progresses. It turns out those matrices are all nonsingular M-matrices, and Ye [30] came up with a new implementation by using the GTH-like algorithm for inverting all nonsingular M-matrices instead of the plain Gaussian elimination. The GTH-like algorithm, due to Alfa, Xue, and Ye [2] (see also [14, p.87]), is a variant of Gaussian elimination, and can guarantee to invert a nonsingular M-matrix, albeit how nearly singular it may be, with high entrywise relative accuracy. Ye's implementation, as a result, was a resounding success – able to compute Φ with high entrywise relative accuracy. The same can be said about the doubling algorithms [6]. The GTH-like algorithm has since been successfully employed in highly accurate solutions of M-matrix Riccati equations [14,18,19,24,26–29], among others.