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Gorenstein Hereditary Coalgebras

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Abstract. Let C be a coalgebra. We prove that a right C-comodule is an n-cosyzygy if and only if it is a Gorenstein n-cosyzygy. We study the Gorenstein global dimension of coalgebras and the class of Gorenstein hereditary coalgebras. As a generalization of Gorenstein hereditary coalgebras, we introduce the concept of Gorenstein semihereditary coalgebras and show that a coalgebra C is Gorenstein semihereditary if and only if every finitely cogenerated factor of any injective comodule is Gorenstein injective.

AMS subject classifications: 16T15, 18G25

Key words: Cosyzygy comodule, Gorenstein injective comodule, hereditary coalgebra, Gorenstein hereditary coalgebra

1 Introduction

Nastasescu et al. first studied the global dimension of coalgebras and discussed the class of coalgebras of global dimension less than or equal to 1. The coalgebras in this class are called hereditary coalgebras [10]. In order to generalize the concept of hereditary coalgebras, Zhu and Tong [12] introduced the concept of semihereditary coalgebras which contains all the cosemisimple coalgebras and hereditary coalgebras. Some results that are dual to semihereditary rings are also given. In recent years, Gorenstein homological coalgebra has been researched extensively by many authors (see, for example, [1,4,9,11]). In 1999, Asensio, López-Ramos and Torrecillas [1] introduced the notion of Gorenstein injective comodules and proved the equivalent conditions for a comodule to be a Gorenstein injective comodule over an n-Gorenstein coalgebra. In 2004, Enochs and López-Ramos [4] studied the class of Gorenstein projective comodules and proved that every comodule over a coalgebra has a Gorenstein injective preenvelope. So every C-comodule has a Gorenstein injective resolution over a coalgebra C.

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In a rencent paper, Huang [6] introduced the notion of a (Gorenstein) syzygy module was defined via the (Gorenstein) projective resolution of modules. Analogously, we introduce the concept of Gorenstein cosyzygy comodule via the Gorenstein injective resolution of comodules as follows. For a positive integer n, a right C-comodule M is called a Gorenstien n-cosyzygy comodule (of a right C-comodule N) if there exists an exact sequence $0\to M\to G_0\to G_1\to\cdots\to G_{n-1}\to N\to 0$ with all G_i Gorenstien injective. It is clear that an n-cosyzygy comodule is Gorenstein n-cosyzygy. The main purpose of Section 3 is to prove that a right C-comodule is an n-cosyzygy if and only if it is a Gorenstein n-cosyzygy.

Motivated by the important role of the coalgebras of global dimensions less than or equal to one in several areas of coalgebra, we study the coalgebras of Gorenstein homological dimensions less than or equal to one in Section 4, which we call Gorenstein hereditary coalgebras. Then we introduce the concept of Gorenstein semihereditary coalgebras and show that a coalgebra C is Gorenstein semihereditary if and only if every finitely cogenerated factor of any injective comodule is Gorenstein injective.

2 Preliminaries

Throughout this paper, we fix a field K. Given a K-coalgebra C, we denote by $\mathcal{M}^C$ and $\mathcal{M}^C_f$ the categories of right C-comodules and right C-comodules of finite dimension, respectively. Given two C-comodules M and N, we denote by $\mathrm{Com}_C(M,N)$ the K-vector space of all C-comodule homomorphisms from M to N.

The category $\mathcal{M}^C$ has enough injective objects, but there are not always enough projective objects. Lin [8] introduced the concept of semiperfect coalgebras which guarantees the existence of enough projective objects. Let $\mathcal{I}$ and $\mathcal{P}$ denote the full subcategory determined by the class of injective comodules and the class of projective comodules, respectively. Injective and projective dimension of a right C-comodule M will be denoted by $\mathrm{id}_C M$ and $\mathrm{pd}_C M$, respectively.

Recall from [1] that a right C-comodule M is called Gorenstein injective if there exists an exact sequence

$$\cdots\to E_1\to E_0\to E^0\to E^1\to\cdots$$

of injective right C-comodules with $M=\mathrm{Ker}(E^0\to E^1)$ and such that the functor $\mathrm{Com}_C(E,-)$ leaves it exact for any injective right C-comodule E. The Gorenstein projective comodules are defined dually in [4]. Let $\mathcal{GP}$ be the full subcategory of $\mathcal{M}^C$ of Gorenstein projective comodules and $\mathcal{GI}$ the class of all Gorenstein injective comodules.

For a coalgebra C, Enochs has proved that every C-comodule has a Gorenstein injective preenvelope [4]. It follows that every C-comodule M has a Gorenstein injective resolution. Let $M\in\mathcal{M}^C$. The Gorenstein injective dimension $\mathrm{Gid}_C(M)$ of M is defined to be the minimal natural number $n\geq 0$ if there exists an exact resolution

$$0\to M\to I^0\to\cdots\to I^n\to 0,$$