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# Equivalent Relations with the J. L. Lions Lemma in a Variable Exponent Sobolev Space and Their Applications

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**Abstract.** We consider the equivalent conditions with $W^{-m,p(\cdot)}$-version of the J. L. Lions Lemma, where $p(\cdot)$ is a variable exponent satisfying some condition. As applications with $m=0$, we first derive the Korn inequality and furthermore, we consider the relation to other fundamental results. One of the purpose of this paper is an application to the existence of a weak solution for the Maxwell-Stokes type problem.

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**Key words**: J. L. Lions Lemma, de Rham Theorem, Maxwell-Stokes type problem, variable exponent Sobolev spaces, multiply connected domain with holes.

## 1 Introduction

Assume that $\Omega$ is a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary $\Gamma$ and $\Omega$ is locally on the same side of $\Gamma$. The classical J. L. Lions Lemma asserts that any distribution in the space of $H^{-1}(\Omega)$ with its gradient (in the distribution sense) belonging to $\boldsymbol{H}^{-1}(\Omega)$ is a function in $L^2(\Omega)$. To various results drawn from the J. L. Lions Lemma, see, for example, Boyer and Fabrie [11] and Ciarlet [12].

Amrouche *et al.* [2] derived the equivalent conditions with the J. L. Lions Lemma. The conditions are the classical and the general J. L. Lions Lemma, the Nečas inequality, the coarse version of the de Rham Theorem, the surjectivity of the operator div : $\boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ and a specific "approximation lemma" which is the key lemma for the proof of equivalence. Some of these equivalent properties can be given as a "direct" proof.

However, these equivalent conditions of $L^2$-version of the J. L. Lions Lemma are insufficient for considering the Maxwell-Stokes type system containing $p$-curlcurl operator.

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Thus it is important for us to improve the result to the $L^p$-version of the equivalent relations with the J. L. Lions Lemma. For this purpose, we improved the results of [2] to the $L^p$-version of the equivalence relation, and we applied the result to the existence of a weak solution to the Maxwell-Stokes type problem in the previous paper Aramaki [5].

Our purposes of this paper is to show the $W^{-m.p(\cdot)}$-version of the equivalent relations of the J. L. Lions Lemma and then to consider the existence of a weak solution to the Maxwell-Stokes type equation containing $p(\cdot)$-curlcurl operators which appears to be new to the best of our knowledge.

We assume that the second Betti number is equal to $I$, that is, $\Gamma$ has $(I+1)$-connected components $\Gamma_1,\cdots,\Gamma_I,\Gamma_{I+1}$ with $\Gamma_{I+1}$ being the boundary of the only unbounded connected component of $\mathbb{R}^3\setminus\overline{\Omega}$. We consider the following Maxwell-Stokes type system. For given $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfying $\boldsymbol{g}\times\boldsymbol{n}=\boldsymbol{0}$ on $\Gamma$, find $(\boldsymbol{u},\pi)$ such that

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}]+\boldsymbol{\nabla}\pi=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}=\boldsymbol{g} & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & i=1,\cdots,I, \end{cases} \tag{1.1}$$

where $S(x,t)$ is a function satisfying some structure conditions depending on a variable exponent, $\langle \boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}d\sigma$, $d\sigma$ is the surface area and $\boldsymbol{n}$ is the outer unit normal vector to $\Gamma$. The existence depends deeply on the nonlinearity of the equation and the shape of the domain. We allow the domain $\Omega$ to be a multiply connected domain with holes. To show the existence, we apply the equivalent conditions with the J. L. Lions Lemma with $m=0$, in particular, the coarse version of the de Rham Theorem. For example, since $L^{p(\cdot)}$-version of the Nečas inequality can be found in Diening *et al.* [13, Theorem 14.3.18], the de Rham Theorem in $L^{p(\cdot)}$-version is correct. For the Maxwell type system containing $p$-curlcurl operator, see Miranda *et al.* [20,21] and Aramaki [7], and for the Maxwell-Stokes type system, see Pan [22] and Aramaki [8] and references therein.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we derive $W^{-m,p(\cdot)}$-version of the J. L. Lions Lemma and its equivalent relations. In Section 4, we consider the Korn inequality. In Section 5, we derive a relation between the J. L. Lions Lemma and a simplified version of the de Rham Theorem. In Section 6, we consider a relation between the J. L. Lions Lemma and a weak version of the Poincaré Lemma. Finally, in Section 7, we give an application on the existence of a weak solution to the Maxwell-Stokes type problem. In the appendix A, we give a proof of the basic theorem (Theorem 7.1).

## 2 Preliminaries

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any normed space $B$, we denote $B^d$ by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard