Orthogonal Exponentials of Planar Self-Affine Measures with Four-Element Digit Set

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Abstract. Let $\mu_{M,D}$ be a self-affine measure generated by an expanding real matrix $M = \begin{pmatrix} a & e \\ f & b \end{pmatrix}$ and the digit set $D = \{(0,0)^t,(1,0)^t,(0,1)^t,(1,1)^t\}$. In this paper, we consider that when does $L^2(\mu_{M,D})$ admit an infinite orthogonal set of exponential functions? Moreover, we obtain that if $e = f = 0$ and $a, b \in \{p, q \in 2\mathbb{Z} + 1\}$, then there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best possible.

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1 Introduction

Let $\mu$ be a compactly supported Borel probability measure on $\mathbb{R}^n$. $\mu$ is called a spectral measure if there exists a discreet set $\Lambda \subseteq \mathbb{R}^n$ such that the collection of exponential functions $E(\Lambda) := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set $\Lambda$ is then called a spectrum for $\mu$.

The existence of a spectrum for $\mu$ is a basic problem in harmonic analysis, it was initiated by Fuglede in his seminal paper [10]. The first singularly continuous spectral measure was discovered by Jorgensen and Pedersen [12]. This surprising discovery has received a lot of attention, and the research on the spectrality of self-affine measures has become an interesting topic. Also, new spectral measures were found in [2, 8, 9, 13, 15, 16] and references cited therein.

For a more general Bernoulli convolution $\mu_{\rho,N}$, Hu and Lau [11] showed that the necessary and sufficient condition that $L^2(\mu_{\rho,2})(0 < \rho < 1)$ contains an infinite orthogonal

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set of exponential functions is that \((\frac{p}{q})^t\) for some positive integers \(p,q,r\), with \(p,q\) being odd and even, respectively. Recently, the spectral problem of Bernoulli convolution was solved by Dai [1]. He showed that \(\mu_{p,2}\) is a spectral measure if and only if the contraction rate \(\rho\) is the reciprocal of an even integer. These results are generalized further to the N-Bernoulli measures [3,5,22].

Unlike the one dimensional case, there are few results in higher dimension. Deng and Lau [6] proved that if \(M=\text{diag}(p,\rho)\) \(|\rho|>1\), with \(D=\{(0,0)^t,(1,0)^t,(0,1)^t\}\), then \(L^2(\mu_{p,D})\) contains an infinite orthogonal set of exponential functions if and only if \(|\rho|=(\frac{p}{q})^{\frac{1}{2^r}}\) for \(p,q,r_\in\mathbb{N}\) with \(\gcd(p,3q)=1\) and \(\mu_{M,D}\) is a spectral measure if and only if \(\rho\in\mathbb{Z}\). Recently, for \(M=\text{diag}(a,b)(a,b>1)\) and above \(D\), Dai, Fu and Yan [4] obtained some more general results.

For non-spectral problem of self-affine measure, Duttan and Jorgensen [7] proved that if \(M=\text{diag}(p,p)\), with \(p\in\mathbb{Z}\setminus\mathbb{Z}\) and \(D=\{(0,0)^t,(1,0)^t,(0,1)^t\}\), then there are no 4 mutually orthogonal exponential functions in \(L^2(\mu_{M,D})\). Following this discovery, the theory of non-spectral measures has been extensively studied, The readers may see [17–19] and references cited therein.

The question of existence of orthogonal families of complex exponentials was raised, and answered, first for \(\text{dim}=1\) in the case of families of Cantor constructions [12]. It turns out that, when an affine fractal is specified as a self-similar measure \(\mu\), then the answer to the question of existence of an orthogonal Fourier basis (ONB) in \(L^2(\mu)\) is sensitive to choice of scaling numbers. For \(\text{dim}=2\), possible Fourier bases exists when the scaling matrix satisfies specific algebraic conditions. The present paper [4,6,7] addresses this question for planar constructions of such measures \(\mu\). For the planar cases, there is a host of additional issues entering into the structure of Fourier bases in \(L^2(\mu)\). This paper concerned with the nature of the set of Fourier frequencies, in the case when a Fourier ONB exists. Let \(\mu_{M,D}\) be a class of self-affine measures which satisfies

\[
\mu_{M,D}(\cdot) = \frac{1}{|D|} \sum_{d \in D} \mu_{M,D}(M(\cdot) - d), \tag{1.1}
\]

where \(M=\begin{pmatrix} a & e \\ f & b \end{pmatrix}\) with \(|\text{det}(M)|>1\), is an expanding matrix, and

\(D=\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \).

Our main results are the following three theorems.

**Theorem 1.1.** If \(e,f \neq 0\) and \(a+b=0\), then \(L^2(\mu_{M,D})\) admits an infinite orthogonal set of exponential functions if and only if \(\text{det}(M)\) is in the set \(\{ \pm (\frac{p}{q})^{\frac{1}{2^r}} : p \in 2\mathbb{Z}^+, q \in 2\mathbb{Z}^+, r \in \mathbb{Z}^+ \}\).

**Theorem 1.2.** If \(e=f=0\), i.e.\(M=\text{diag}(a,b)\), then \(L^2(\mu_{M,D})\) admits an infinite orthogonal set of exponential functions if and only if there exist a number of \(a,b\) that are in the set \(\{ \pm (\frac{p}{q})^{\frac{1}{2^r}} : p \in 2\mathbb{Z}^+, q \in 2\mathbb{Z}^+, r \in \mathbb{Z}^+ \}\).