J. Math. Study
doi: 10.4208/jms.v55n1.22.07

Vol. **55**, No. 1, pp. 84-94
March 2022

Repdigits Base b as Difference of Two Fibonacci Numbers

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Received February 21, 2020; Accepted May 10, 2021;
Published online January 18, 2022.

Abstract. In this paper, we find all repdigits expressible as difference of two Fibonacci numbers in base b for $2 \leq b \leq 10$. The largest repdigits in base b, which can be written as difference of two Fibonacci numbers are

$$\begin{aligned} &F_9 - F_4 = 34 - 3 = 31 = (11111)_2, && F_{14} - F_7 = 377 - 13 = 364 = (111111)_3, \\ &F_{14} - F_7 = 377 - 13 = 364 = (222)_4, && F_9 - F_4 = 34 - 3 = 31 = (111)_5, \\ &F_{11} - F_4 = 89 - 3 = 86 = (222)_6, && F_{13} - F_5 = 233 - 5 = 228 = (444)_7, \\ &F_{10} - F_2 = 55 - 1 = 54 = (66)_8, && F_{14} - F_7 = 377 - 13 = 364 = (444)_9, \end{aligned}$$

and

$$F_{15} - F_{10} = 610 - 55 = 555 = (555)_{10}.$$

As a result, it is shown that the largest Fibonacci number which can be written as a sum of a repdigit and a Fibonacci number is $F_{15} = 610 = 555 + 55 = 555 + F_{10}$.

AMS subject classifications: 11B39, 11J86, 11D61

Key words: Fibonacci numbers, repdigit, Diophantine equations, linear forms in logarithms.

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1 Introduction

Let (F_n) and (L_n) be the sequences of Fibonacci and Lucas numbers given by $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, respectively. Binet formulas for these numbers are $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, which are the roots of the characteristic equation $x^2 - x - 1 = 0$. It can be seen that $1 < \alpha < 2$, $-1 < \beta < 0$

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and $\alpha\beta=-1$. For more about Fibonacci and Lucas numbers with their applications, one can see [9]. The relation between Fibonacci number F_n and α are given by

$$\alpha^{n-2}\leq F_n\leq\alpha^{n-1} \tag{1.1}$$

for $n\geq 1$. The inequality (1.1) can be proved by induction. A repdigit in base b is a positive integer N whose digits are all equal. If $b=10$, we say that N is a repdigit. Recently, the problem of finding the repdigits in the second-order linear recurrence sequences has been of interest to mathematicians. In [10], Luca has found all repdigits in the Fibonacci and Lucas sequences. The largest repdigits in Fibonacci and Lucas sequences are $F_{10}=55$ and $L_5=11$. Luca [11] also found all repdigits which are sums of three Fibonacci numbers. In [6], Erduvan and Keskin showed that if F_mF_n is a repdigit in base 10, then $F_mF_n\in\{1,2,3,4,5,6,8,9,55\}$. Later Keskin, Erduvan and Şiar [7] have found that if F_mF_n is a repdigit in base b, and has at least two digits, then $F_mF_n\in\{3,4,5,6,8,9,10,13,15,16,21,24,26,40,42,63,170,273\}$, where $b=2,3,4,5,6,7,8,9$ and $2\leq m\leq n$. Motivated by the above studies, here, we consider the Diophantine equation

$$F_n-F_m=N=\frac{d\cdot(b^k-1)}{b-1}, \tag{1.2}$$

where $2\leq b\leq 10$, $2\leq m<n$, $1\leq d\leq b-1$ and $k\geq 2$. Since the values F_1 and F_2 are the same, we start this equation from $m=2$ instead of $m=1$. In Section 2, we introduce necessary lemmas and theorems. Then in Section 3, we prove our main results (Theorem 3.1) on the solutions to Eq. (1.2).

2 Auxiliary results

In order to solve Diophantine equations of the form (1.2), we use Baker's theory for lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are very important in effectively solving of Diophantine equations, we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with the minimal polynomial

$$a_0x^d+a_1x^{d-1}+\ldots+a_d=a_0\prod_{i=1}^{d}\left(x-\eta^{(i)}\right)\in\mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0>0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta)=\frac{1}{d}\left(\log a_0+\sum_{i=1}^{d}\log\left(\max\left\{|\eta^{(i)}|,1\right\}\right)\right) \tag{2.1}$$

is called the logarithmic height of η. If $\eta=a/b$ is a rational number with $\gcd(a,b)=1$ and $b\geq 1$, then $h(\eta)=\log(\max\{|a|,b\})$.