J. Math. Study
doi: 10.4208/jms.v55n1.22.08

Vol. **55**, No. 1, pp. 95-108
March 2022

A Kind of Integral Representation on Complex Manifold

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Received February 13, 2021; Accepted November 7, 2021;
Published online January 13, 2022.

Abstract. In this paper, by using the Hermitian metric and Chern connection, we study the case of a strictly pseudoconvex domain G with non-smooth boundaries in a complex manifold. By constructing a new integral kernel, we obtain a new Koppelman–Leray–Norguet formula of type (p,q) on G, and get the continuous solutions of $\bar{\partial}$–equations on G under a suitable condition. The new formula doesn't involve integrals on the boundary, thus one can avoid complex estimations of the boundary integrals, and the density of integral may be not defined on the boundary but only in the domain. As some applications, we discuss the Koppelman–Leray–Norguet formula of type (p,q) for general strictly pseudoconvex polyhedrons (unnecessarily non-degenerate) on Stein manifolds, also get the continuous solutions of $\bar{\partial}$–equations under a suitable condition.

AMS subject classifications: 32A26, 32Q99, 32T15

Key words: Complex manifold, strictly pseudoconvex domain, non-smooth boundary, Koppelman–Leray–Norguet formula, $\bar{\partial}$–equation

1 Introduction

As early as 1831, Cauchy found the famous Cauchy integral formula, which named by his name. Many mathematicians were aware of the importance of integral representation in complex analysis. Later, the integral representation method gradually became one of the main methods of complex analysis in several variables, because one of its main virtues is that it is easy to estimate like the Cauchy integral formula in one complex variable. It is well known that the integral representations and their applications for $(0,q)$ differential form in $\mathbf{C}^n$ have been deeply studied(see, for instance [1–10]). But the research for

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integral representations on complex manifolds began in 1980s. Most of the results, so far, have been concerned with Stein manifolds(see, for instance [4,5,11-13]). In the early 1990s, Berndtsson B [14] studied the theory of integral representations on general complex manifolds and gained in a quite general integral kernel under a suitable condition. Using this kernel, the Koppelman formula on complex manifolds was obtained. Based on it, Zhong T [15] got the Koppelman–Leray–Norguet formula of type (p,q) on a bounded domain G with piecewise C^1 smooth boundaries in a complex manifold, and gave the continuous solutions of $\bar{\partial}$–equations on G under a suitable condition.

In this paper, by using the Hermitian metric and Chern connection, the case of a strictly pseudoconvex domain G with non-smooth boundaries in a complex manifold is studied. By constructing a new integral kernel, a new Koppelman–Leray–Norguet formula of type (p,q) on G is obtained, and the continuous solution of $\bar{\partial}$–equations on G under a suitable condition is given. The new formula doesn't involve integrals on the boundary, thus complex estimations of the boundary integrals can be avoided, and the density of integral may be not defined on the boundary but only in the domain. As an application, $\mathbf{C}^n$ and Stein manifold are taken as examples to discuss the relationship between the conclusion of this paper and the corresponding conclusion in [4]. The Koppelman–Leray–Norguet formula of type (p,q) for general strictly pseudoconvex polyhedrons with unnecessarily non-degenerate on Stein manifolds is also discussed, and the continuous solution of $\bar{\partial}$–equations under a suitable condition is given, it implies the corresponding result in paper [13].

2 Basic knowledge and lemma

Let M be a complex manifold, $X = M\times M$, E is supposed to be a holomorphic vector bundle of rank n over X, and η is a holomorphic section to E such that

$$\{\eta=0\}=Y=\{(\zeta,z)\in X|\zeta=z\}.$$

Let ξ be any smooth section to E^* that is a dual bundle of E, which is admissible for η, i.e. For any compact set $B\subseteq X$, we have

$$|\xi|\le c_B|\eta| \quad \text{and} \quad |\langle\xi,\eta\rangle|\ge C_B|\eta|^2,$$

where c_B and C_B are constants only concerned with B, e.g. ξ is dual vector of η on some measurement, then ξ is admissible for η. We consider extensional equation

$$dK=[Y]-C_n[\Theta], \tag{2.1}$$

where Θ is a formal curvature of some connections for E, $C_n[\Theta]$ is the n^{th} Chern form of Θ. Berndtsson B obtained explicit solution of (2.1)(name it Berndtsson's kernel)

$$K[\xi,\eta](z,\zeta)\Lambda=\frac{\xi\wedge D\eta}{n!(2\pi i)^n}\sum_{k=0}^{n-1}\begin{pmatrix} n\\ k\end{pmatrix}(-1)^k\frac{(D^*\xi\wedge D\eta)^{n-k-1}}{\langle\xi,\eta\rangle^{n-k}}\wedge\widetilde{\Theta}^k, \tag{2.2}$$