

## Protected Branches in Ordered Trees

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**Abstract.** In this paper, we consider the class of ordered trees and its two subclasses, bushes and planted trees, which consist of the ordered trees with root degree at least 2 and with root degree 1 respectively. In these three classes, we study the number of trees of size  $n$  with  $k$  protected (resp. unprotected) branches, and the total number of branches (resp. protected branches, unprotected branches) among all trees of size  $n$ . The explicit formulas as well as the generating functions are obtained. Furthermore, we find that, in each class, as  $n$  goes to infinity, the proportion of protected branches among all branches in all trees of size  $n$  approaches  $1/3$ .

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**Key words:** ordered tree, bush, planted tree, protected branch, unprotected branch, Catalan number, generating function.

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## 1 Introduction

An ordered tree is defined recursively as having a root and an ordered set of subtrees [14, 21]. We will draw ordered trees with the root on the top level, the root being connected with the roots of its subtrees by line segments, called edges. The size of a tree is defined to be the number of edges. For each vertex  $v$ , the number of subtrees rooted at  $v$  is defined as the degree of  $v$ . In the graph theory, it is also named as outdegree. There are many different definitions for bush [8, 11, 12]. In this paper, we use the definition in [8]. A bush is an ordered tree in which the degree of the root is at least 2, while a planted tree is an ordered tree with root degree 1. The 14 ordered trees of size 4 are shown in Figure 1, in which the first 5 trees are planted trees and the remaining 9 trees are bushes.

A vertex of degree zero is called a leaf. A vertex of positive degree is called an internal node. A protected point is a vertex which is not a leaf and which is not distance 1 from a leaf. Cheon and Shapiro [5] started the study of protected points in ordered trees, and

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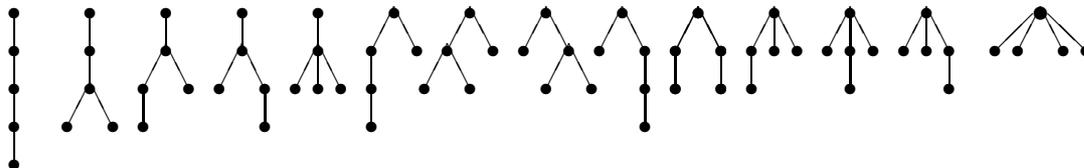


Figure 1: The 14 ordered trees of size 4 in which the last 9 are bushes.

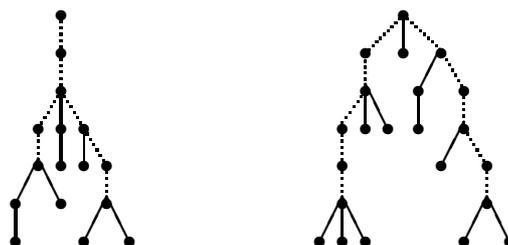


Figure 2: A planted ordered tree and a bush.

they showed that the average portion of protected points in ordered trees with  $n$  edges approaches  $1/6$  as  $n$  goes to  $\infty$ . Since this pioneering paper, a large number of extensions have been studied [3,4,7,13,15,16]. We will focus on protected branches in ordered trees in this paper. An internal node of degree at least 2 is called a branch node. A tree with no branch nodes is called a path. By a branch we mean a path connecting either the root and a nearest branch node, or two nearest branch nodes, or a leaf and the nearest branch node. Riordan [18] enumerated the plane trees by number of branches and endpoints. Deutsch [12] introduced a new decomposition of ordered trees by branches, and using this decomposition he enumerated the ordered trees with prescribed root degrees, node degrees, and branch lengths.

We define a protected branch of a ordered tree is a branch which does not contain a leaf, and an unprotected branch is a branch ending at a leaf. For instance, a planted tree with 4 protected branches and a bush with 5 protected branches are displayed in Figure 2, in which the dashed branches are all protected branches.

In this paper, we will enumerate the number of ordered trees (resp. bushes, planted trees) of size  $n$  with  $k$  protected (resp. unprotected) branches, and the total number of branches (resp. protected branches, unprotected branches) among all ordered trees (resp. bushes, planted trees) of size  $n$ . In Section 2, we compute the numbers of branches in ordered trees, bushes, and planted trees. In Section 3, the enumerations of protected branches will be discussed. We will show that, as  $n$  goes to infinity, the average proportion of protected branches among all branches of ordered trees (resp. bushes, planted trees) of size  $n$  approaches  $1/3$ . In Section 4, the enumerations of unprotected branches will be considered. As by-products, we obtain three new combinatorial interpretations