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Besov Spaces with General Weights

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Abstract. We introduce Besov spaces with general smoothness. These spaces unify and generalize the classical Besov spaces. We establish the φ-transform characterization of these spaces in the sense of Frazier and Jawerth and we prove their Sobolev embeddings. We establish the smooth atomic, molecular and wavelet decomposition of these function spaces. A characterization of these function spaces in terms of the difference relations is given.

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1 Introduction

Function spaces have been a central topic in modern analysis, and are now of increasing applications in many fields of mathematics especially harmonic analysis and partial differential equations. The most known general scales of function spaces are the scales of Besov spaces and Triebel-Lizorkin spaces and it is known that they cover many well-known classical function spaces such as Hölder-Zygmund spaces, Hardy spaces and Sobolev spaces. For more details one can refer to Triebel's books [58–60].

In recent years many researchers have modified the classical spaces and have generalized the classical results to these modified ones. For example: Function spaces of generalized smoothness. These types of function spaces have been introduced by several authors. We refer, for instance, to Bownik [8], Cobos and Fernandez [14], Goldman [30] and [31], and Kalyabin [38]; see also Kalyabin and Lizorkin [39].

The theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the study of trace spaces on fractals, see Edmunds and Triebel [22, 23], where they introduced the spaces $B^{s,\Psi}_{p,q}$, where Ψ is a so-called admissible function, typically of log-type near 0. For a complete treatment of these spaces we refer the readers to the work of Moura [46].

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Besov [2–4] defined function spaces of variable smoothness and obtained their characterizations by differences, interpolation, embeddings and extension. Such spaces are a special case of the so-called 2-microlocal function spaces. The concept of 2-microlocal analysis, or 2-microlocal function spaces, is due to Bony [5]. These type of function spaces have been studied in detail in [40]. We mention the papers [41, 42] and references given therein.

More general function spaces of generalized smoothness can be found in Farkas and Leopold [24], and reference therein.

Tyulenev has introduced in [62] a new family of Besov spaces of variable smoothness which cover many classes of Besov spaces, where the norm on these spaces was defined with the help of classical differences.

Based on this weighted class and the Fourier-analytical methods we introduce Besov spaces of variable smoothness consisting of tempered distributions and present their essential properties such as the φ-transforms characterization, Sobolev embeddings, atomic, molecular and wavelet decompositions.

The paper is organized as follows. First we give some preliminaries where we fix some notations and recall some basic facts on the Muckenhoupt classes and the weighted class of Tyulenev. Also we give some key technical lemmas needed in the proofs of the main statements. We then define the Besov spaces as follows. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions φ on $\mathbb{R}^n$, i.e., φ is infinitely differentiable and

$$\big\|\varphi|\mathcal{S}_M\big\| := \sup_{\beta\in\mathbb{N}_0^n, |\beta|\le M} \sup_{x\in\mathbb{R}^n} |\partial^\beta \varphi(x)|(1+|x|)^{n+M+|\beta|} < \infty$$

for all $M\in\mathbb{N}$. Select a Schwartz function φ such that

$$\operatorname{supp}\mathcal{F}\varphi \subset \{\xi : \frac{1}{2}\le|\xi|\le 2\}$$

and

$$|\mathcal{F}\varphi(\xi)|\ge c, \qquad \text{if} \quad \frac{3}{5}\le|\xi|\le\frac{5}{3},$$

where $c>0$ and we put $\varphi_k = 2^{kn}\varphi(2^k\cdot)$, $k\in\mathbb{Z}$. Here $\mathcal{F}\varphi$ denotes the Fourier transform of φ, defined by

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \qquad \xi\in\mathbb{R}^n.$$

Let

$$\mathcal{S}_\infty(\mathbb{R}^n) := \Big\{\varphi\in\mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\beta\varphi(x)dx = 0 \text{ for all multi-indices } \beta\in\mathbb{N}_0^n\Big\}.$$

Following Triebel [58], we consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Thus, $\mathcal{S}_\infty(\mathbb{R}^n)$ is a complete metric space. Equivalently, $\mathcal{S}_\infty(\mathbb{R}^n)$ can be defined as a collection of all $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that semi-norms

$$\|\varphi\|_M := \sup_{|\beta|\le M} \sup_{\xi\in\mathbb{R}^n} |\partial^\beta\varphi(\xi)|\big(|\xi|^M + |\xi|^{-M}\big) < \infty$$