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# Lower Bounds on the Number of Cyclic Subgroups in Finite Non-Cyclic Nilpotent Groups

Wei Meng[1,*] and Jiakuan Lu[2]

[1] *School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China;*
[2] *School of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China.*

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**Abstract.** Let $G$ be a finite group and $\mathfrak{c}(G)$ denote the number of cyclic subgroups of $G$. It is known that the minimal value of $\mathfrak{c}$ on the set of groups of order $n$, where $n$ is a positive integer, will occur at the cyclic group $Z_n$. In this paper, for non-cyclic nilpotent groups $G$ of order $n$, the lower bounds of $\mathfrak{c}(G)$ are established.

**AMS subject classifications**: 20D10, 20D20

**Key words**: $p$-groups, cyclic subgroups, Nilpotent groups.

## 1 Introduction

Throughout this paper all groups are finite. For a group $G$ of order $n$, let $\mathfrak{c}(G)$ denote the number of cyclic subgroups of $G$ and $d(n)$ denote the number of divisors of $n$. A well-known result on group theory says that a cyclic group of order $n$ has a unique subgroup of order $d$, for any divisor of $n$, so a cyclic group of order $n$ has exactly $d(n)$ (necessarily cyclic) subgroups. Richard [14] proved that $\mathfrak{c}(G)\geq d(n)$, with equality if and only if $G$ is a cyclic group. Another basic result of group theory states that $\mathfrak{c}(G)=|G|$ if and only if $G$ is an elementary abelian 2-group. Tărnăuceanu [16, 17] described the finite groups with $\mathfrak{c}(G)=|G|-r$ ($r=1,2$). Regarding the results about $\mathfrak{c}(G)=|G|-r$. Belshoff, Dillstrom and Reid [2, 3] established a more remarkable bound. They showed that $|G|\leq 8r$. Cocke and Jensen [4] proved that if $G$ is not a 2-group then $|G|\leq 6r$. Jafari and Madadi [9] proved that for any a divisor $m$ of $|G|$, $G$ has at least $d(m)$ cyclic subgroups whose orders divide $m$. Garonzi and Lima [5] studied the function $\alpha(G)=\frac{\mathfrak{c}(G)}{|G|}$. They explored basic properties of $\alpha(G)$ and pointed out a connection with the probability of commutation.

*Corresponding author. *Email addresses:* mlwhappyhappy@163.com (Meng W), jklu@gxnu.edu.cn (Lu J)

Let $\mathfrak{s}(G)$ denote the number of subgroups of $G$. It's well-known that if $G$ is a $p$-group of order $p^n$, then $\mathfrak{s}(G)\le\mathfrak{s}(Z_p^n)$. Qu [13] proved that if $p$ is odd and $G$ is non-elementary abelian $p$-group, then

$$\mathfrak{s}(G)\le\mathfrak{s}(M_p\times Z_p^{n-3}),$$

where $M_p=\langle a,b\,|\,a^p=b^p=c^p=1,[a,b]=c,[a,c]=[b,c]=1\rangle$. Tărnăuceanu [18] showed that if $G$ is a non-elementary abelian 2-group of order $2^n$, then

$$\mathfrak{s}(G)\le\mathfrak{s}(D_8\times Z_2^{n-3}).$$

Aivazidis and Müller [1] determined the structure of those finite non-cyclic $p$-groups whose number of subgroups is minimal. Recently, we [12] generalized the results of Aivazidis and Müller on all finite non-cyclic nilpotent groups.

In the light of above investigations, it is a natural question that to ask for a given order which non-cyclic groups have the minimal number of cyclic subgroups. In this paper, this question is answered among all non-cyclic nilpotent groups. In fact, we obtain the lower bounds of $\mathfrak{c}(G)$, where $G$ is a non-cyclic nilpotent of order $n$. Our main results are the following theorems.

**Theorem 1.1.** *Let $p$ be a prime, $G$ a non-cyclic $p$-group of order $p^n$.*

(1) *If $p^n=2^3$, then $\mathfrak{c}(G)\ge 5$, with equality if and only if $G\cong Q_8$.*

(2) *If $p^n\ne 2^3$, then $\mathfrak{c}(G)\ge(n-1)p+2$, with equality if and only if $G\cong Z_{p^{n-1}}\times Z_p$, $M_{p^n}$ or $Q_{16}$.*

**Theorem 1.2.** *Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a positive integer and $s=\min\{i\in\{1,\cdots,k\}\,|\,\alpha_i>1\}$, where $p_1<p_2<\cdots<p_k$ are distinct primes. Suppose $G$ is a non-cyclic nilpotent group of order $n$, then there exists a suitable $q\in\pi(n)$, such that $Q$ is non-cyclic and $p_s\le q\le 3p_s-2$, where $Q\in Syl_q(G)$. Furthermore,*

(1) *If $q^\lambda=2^3$, then $\mathfrak{c}(G)\ge 5\cdot d(\frac{n}{8})$, with equality if and only if $G\cong Q_8\times Z_{\frac{n}{8}}$.*

(2) *If $q^\lambda\ne 2^3$, then $\mathfrak{c}(G)\ge[(\lambda-1)q+2]\cdot d(\frac{n}{q^\lambda})$, with equality if and only if $G\cong Z_q\times Z_{\frac{n}{q}}$, $M_{q^\lambda}\times Z_{\frac{n}{q^\lambda}}$ or $Q_{16}$.*

All unexplained notations and terminologies are standard and can be found in [6,8,15]. In addition, $\pi(n)$, the set of the prime divisors of $n$; $Z_n$, the cyclic group of order $n$; $Q_{2^n}$, the generalized quaternion of order $2^n$; $Z_p^n$, the elementary abelian group of order $p^n$; $M_{p^\lambda}=\langle a,b\,|\,a^{p^{\lambda-1}}=b^p=1,a^b=a^{1+p^{\lambda-2}}\rangle$. $A\times B$ means a direct product of $A$ and $B$.

## 2 Preliminaries

**Lemma 2.1.** ([7]) *Let $p$ be an odd prime, $G$ a $p$-group of order $p^n$ with $exp(G)=p^{n-\alpha}(n\ge 3)$. If $\alpha\ge 1$, then $\mathfrak{c}_k(G)\equiv 0$ mod $p$, where $2\le k\le n-\alpha$.*