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Regularity for p-Harmonic Functions in the Grušin Plane

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Abstract. Let $X=\{X_1,X_2\}$ be the orthogonal complement of a Cartan subalgebra in the Grušin plane, whose orthonormal basis is formed by the vector fields X_1 and X_2. When $1<p<\infty$, we prove that weak solutions u to the degenerate subelliptic p-Laplacian equation

$$\triangle_{X,p}u(z)=\sum_{i=1}^{2}X_i(|Xu|^{p-2}X_iu)=0$$

have the $C^{0,1}_{\rm loc}$, $C^{1,\alpha}_{\rm loc}$ and $W^{2,2}_{X,{\rm loc}}$-regularities.

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1 Introduction

Consider the following elliptic equation in divergence form

$$\sum_{i=1}^{2}X_i(a_i(Xu))=0 \tag{1.1}$$

in the Grušin plane, namely, $\mathbb{R}^2$ endowed with a vector field $X=\{X_1,X_2\}$, where $X_1=\partial_x$ and $X_2=x\partial_y$. See Section 2 for more geometries and properties of the Grušin plane. In what follows, we always suppose that $a\in C^2(\mathbb{R}^2,\mathbb{R})$ satisfies the following growth and ellipticity condition:

$$\begin{cases} \sum\limits_{i,j=1}^{2} a_{ij}(\xi)\eta_i\eta_j \ge l_0(\delta+|\xi|^2)^{\frac{p-2}{2}}|\eta|^2, \\ \sum\limits_{i,j=1}^{2}|a_{ij}(\xi)|\le L(\delta+|\xi|^2)^{\frac{p-2}{2}} \quad \text{and} \quad |a_i(\xi)|\le L(\delta+|\xi|^2)^{\frac{p-2}{2}}|\xi| \end{cases} \tag{1.2}$$

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for all $\xi,\eta\in\mathbb{R}^2$, where $0\le\delta\le 1$, $p>1$, and $0<l_0<L$. In this paper, for any function $a\in C^2(\mathbb{R}^2,\mathbb{R})$, we denote by $a_i(\xi):=\frac{\partial a(\xi)}{\partial\xi_i}$ the Euclidean partial derivative of a for any $\xi\in\mathbb{R}^2$ and $1\le i\le 2$ and by $a_{ij}(\xi):=\frac{\partial^2 a(\xi)}{\partial\xi_i\partial\xi_j}$ the second order Euclidean partial derivative of a for $1\le i,j\le 2$.

Given a bounded domain $\Omega\subset\mathbb{R}^2$, we say that $u\in W_X^{1,p}(\Omega)$ is a weak solution to (1.1) if

$$\sum_{i=1}^{2}\int_\Omega a_i(Xu)X_i\varphi dz=0,\quad \forall\varphi\in C_0^\infty(\Omega). \tag{1.3}$$

Here $W_X^{1,p}(\Omega)$ is the horizontal Sobolev space, that is, all $v\in L^p(\Omega)$ with the distributional horizontal derivative $Xv\in L^p(\Omega)$. Note that (1.1) is corresponds to the Euler-Lagrange equation for the energy functional $I=\int_\Omega a(Xu)dz$. It is well known that the local minimizer of I is equivalent to the weak solution to (1.1), see [35, Section2.2]. In the typical case $a(\xi)=(\delta+|\xi|^2)^{\frac{p}{2}}$, the corresponding Euler-Lagrange equation is the non-degenerate p-Laplacian equation

$$\sum_{i=1}^{2}X_i((\delta+|Xu|^2)^{\frac{p-2}{2}}X_iu)=0\quad\text{if}\quad\delta>0, \tag{1.4}$$

and the p-Laplacian equation

$$\sum_{i=1}^{2}X_i(|Xu|^{p-2}X_iu)=0\quad\text{if}\quad\delta=0. \tag{1.5}$$

In particular, weak solutions to (1.5) are called as p-harmonic functions.

In this paper, we mainly focus on the $C^{0,1}_{\rm loc}$, $C^{1,\alpha}_{\rm loc}$ and $W^{2,2}_{X,\rm loc}$-regularities of weak solutions to (1.1) in the Grušin plane. Here for any function $v\in\Omega$, we say that v belongs to $W^{2,2}_{X,\rm loc}(\Omega)$ if $v\in W^{1,2}_{X,\rm loc}(\Omega)$ and its second order distributional horizontal derivative XXv belongs to $L^2_{\rm loc}(\Omega)$, where $XXv=(X_iX_jv)_{1\le i,j\le 2}$. The first result provides Lipschitz regularity of weak solutions.

Theorem 1.1. *Let $1<p<\infty$ and $0\le\delta\le 1$. Assume that $a\in C^2(\mathbb{R}^2,\mathbb{R})$ satisfies the condition* (1.2). *If $u\in W^{1,p}_{X,\rm loc}(\Omega)$ is a weak solution to* (1.1), *in particular, if u is a p-harmonic function, then $Xu\in L^\infty_{\rm loc}(\Omega;\mathbb{R}^2)$. Moreover, for any ball $B_r\subset\Omega$, we have*

$$\sup_{B_{r/2}}|Xu|\le C(p,L,l_0)\left(\fint_{B_r}(\delta+|Xu|^2)^{\frac{p}{2}}dz\right)^{\frac{1}{p}}. \tag{1.6}$$

In this paper, we denote by $B_r(z)$ the ball centered at $z\in\mathbb{R}^2$ with radius $r>0$ with respect to the Carnot-Carathéodory distance d_X determined by X. For simplicity we use B_r to denote $B_r(z)$ for some z and write $C(a,b,\ldots)$ as a positive constant depending on parameter $a,b,\ldots$, whose value may change line to line.

We further obtain the following $C^{1,\alpha}_{\rm loc}$-regularity of weak solutions.