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Finite Groups and the Sum of Orders of Their Subgroups

Jiakuan Lu[1,*], Meiqun Kuang[1], Kaixun Wu[2] and Boru Zhang[1]

[1] *School of Mathematics and Statistics, Guangxi Normal University, Guilin 541006, China.*
[2] *School of Science and Technology, Guilin University, Guilin 541006, China.*

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Abstract. Let G be a finite group and $\sigma_1(G)=\frac{1}{|G|}\sum_{H\le G}|H|$. In this paper, we prove that if G is a nonsolvable group and $\sigma_1(G)=\frac{117}{20}$, then $G=A_5$.

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Key words: Finite groups, Solvable groups, subgroup orders.

1 Introduction

In this paper, all groups are assumed to be finite. The notation is standard and follows that of Isaacs [5]. Let G be a group. Followed by [7], we write

$$\sigma_1(G)=\frac{1}{|G|}\sum_{H\le G}|H|.$$

In [4], the authors proved the following theorem, which answered positively to a problem posed by Tărnăuceanu in [7].

Theorem 1.1. *Let G be a group. If $\sigma_1(G)<\frac{117}{20}$, then G is solvable.*

Since $\sigma_1(A_5)=\frac{117}{20}$, the bound in Theorem 1.1 is the best possible. In this paper, using the idea in [4], we prove the following result:

Theorem 1.2. *Let G be a nonsolvable group. If $\sigma_1(G)=\frac{117}{20}$, then $G=A_5$.*

*Corresponding author. *Email address:* `jklu@mailbox.gxnu.edu.cn` (Lu J), `1450223512@qq.com` (Kuang M), `1547049416@qq.com` (Wu K), `brzhangyq@163.com` (Zhang B)

2 Lemmas

Lemma 2.1. *Let G be a group and N be a nontrival proper normal subgroup of G. Then* $\sigma_1(G/N) < \sigma_1(G)$.

Proof. By definition, we have that

$$\sigma_1(G/N) = \frac{1}{|G|}\sum_{N\le H\le G}|H| = \sum_{N\le H\le G}\frac{1}{|G:H|},$$
$$\sigma_1(N) = \frac{1}{|N|}\sum_{H\le N}|H| = |G:N|\sum_{H\le N}\frac{1}{|G:H|}.$$

Therefore,

$$\sigma_1(G) \ge \sigma_1(G/N) + \frac{1}{|G:N|}(\sigma_1(N)-1) > \sigma_1(G/N),$$

as desired. □

Lemma 2.2 ([3]). *If a group G has an abelian maximal subgroup, then G is solvable.*

Lemma 2.3 ([1]). *If a group G has at most 2 conjugacy classes of non-normal maximal subgroups, then G is solvable.*

Lemma 2.4 ([1]). *Let G be a non-solvable group. Then G has three conjugacy classes of maximal subgroups if and only if either* $G/\Phi(G) = PSL(2,7)$ *or* $G/\Phi(G) = PSL(2,2^p)$, *where p is a prime.*

Lemma 2.5 ([4]). *Let* $G = PSL(2,2^p)$, *where* $p \ge 5$ *is a prime. Then*

$$\sum_{H\le G \text{ is not cylic}}|H| \ge p|G|.$$

Lemma 2.6 ([4]). *Let G be a group and* $\mathcal{K}$ *be the conjugacy class containing a self-normalizing subgroup K of G. Then*

$$\sum_{H\in\mathcal{K}}|H| = |G|.$$

Lemma 2.7 ([2]). *Let G be a group.Then*

$$\sum_{H\le G \text{ is cylic}}|H| \ge |G|.$$