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# Comparison Between Conformal Invariants for Conformally Compact Einstein Metrics: Some Counter-Example from the Metrics Developed by Pedersen

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**Abstract.** The study of asymptotically hyperbolic Einstein metric is a rich field in theoretical physics and geometry. Pedersen introduced a family of examples for the dimension 4, and we look in this paper into the sign of some of its conformal invariant, namely renormalized volume and Yamabe-type constant. This brings some insights in the study of conformally compact Einstein manifold, as the comparison of invariants is already common practice.

**AMS subject classifications**: l53C25, 30F45,58C35

**Key words**: Conformally compact Einstein manifolds, Berger sphere at infinity, Renormalized volume, Yamabe-Escobar constant.

## 1 Introduction

The study of conformal invariant for asymptotically hyperbolic manifold has been of great importance in physics since the introduction by Maldacena in [1] of the AdS-CFT correspondence. A great idea is to try to extract information from conformal structure at infinity. An example of such a result is proved by Qing in [6] (see also [9], and another proof from Han-Gursky in [10]) †:

**Theorem 1.1** (Quing {2003}-Han-Gursky {2017})**.** *Let* $(\overline{M}^{n+1}, \partial M^n, g_+)$ *be a conformally compact Poincaré-Einstein manifold of class* $C^2$ *satisfying that either the dimension* $3 \leq n+1 \leq 5$

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†the definition of the Yamabe constant is recalled in section 5 and the one of the Yamabe-Escobar constant is in section 6

*or that the dimension $n+1\geq 6$ and $\overline{M}$ is spin. We note $(\partial M,[\gamma])$ the conformal infinity of our space* ‡*. If the Yamabe constant of the conformal infinity is positive $Y(\partial M,[\gamma])>0$, then the first type of Yamabe-Escobar constant of the whole metric must be positive:*

$$Y^1(M,\partial M,g_+)>0$$

A natural question is then if the converse could be true, and if we could get the same type of result for the renormalized volume instead of the first type of Yamabe-Escobar constant. This paper show that they both are not possible with an explicit calculation with a well known family of metric.

Pedersen introduced in [2] a family of conformally hyperbolic Einstein metrics on the 4-ball with non-isometric conformal infinity (the Berger sphere), which has given a great deal of examples or counter-examples in the study of such Riemannian metrics.

The aim of the present paper is to give a general comprehension of some conformal invariants for these metrics, namely the renormalized volume, the sign of the infinity's Yamabe constant and the sign of the Yamabe-Escobar constant. More precisely, we will prove:

**Theorem 1.2.** *Let $m>-1$ and $g_m$ be the Pedersen metric of parameter $m$,*

$(a)$ *The renormalized volume of this Pedersen metric is:*

$$V(B_1(\mathbb{R}^4),g_m)=\frac{4\pi^2}{3}\left(1-\frac{m^2}{(1+m)^2}\right),$$

*and it is positive for $-\frac{1}{2}\leq m$ and negative otherwise;*

$(b)$ *The infinity's conformal Yamabe constant is positive for $-\frac{3}{4}\leq m$ and negative otherwise;*

$(c)$ *There exists $m_0\leq -\frac{9}{10}$ such that the conformal Yamabe-Escobar constant is positive for $m_0\leq m$ and negative otherwise.*
*Furthermore, $m_0$ is the only solution in $]-1,0[$ of the equation $4\sqrt{-m}=\ln\left(\frac{1+\sqrt{-m}}{1-\sqrt{-m}}\right)$.*

In Section 2, we recall the definition of the metric constructed by Pedersen, before recalling the Riemannian curvature of such metric in an adapted orthonormal frame in 3, which allow us to calculate the $L^2$ norm of the Weyl curvature. Then in Section 4 we prove the point (a), in Section 5 the point (b) and in Section 6 the point (c).

## 2 Definition of the metrics

Let's recall that on $\mathbb{S}^3(\mathbb{R})$ there is a continuous family of Riemannian non locally isometric metrics called Berger metrics on the sphere (in short, we will call Berger metrics

‡we recall that if $\gamma=\overline{g}_{|\partial M}\in\Sigma^2T^*\partial M$ depend on the choice of a compactification, its conformal class $[\gamma]$ does not. So its Yamabe constant, a conformal invariant, does not depend of the choice of such a compactification.