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The Cauchy Problem for the Sixth Order p-Generalized Benney-Luke Equation

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Abstract. We investigate the Cauchy problem for the sixth order p-generalized Benney-Luke equation. The local well-posedness is established in the energy space $\dot{H}^1(\mathbb{R}^n)\cap \dot{H}^3(\mathbb{R}^n)$ for $1\le n\le 10$, by means of the Sobolev multiplication law and the contraction mapping principle. Moreover, we establish the energy identity of solutions and provide the sufficient conditions of the global existence of solutions by analyzing the properties of the energy functional.

AMS subject classifications: 35L30, 76B15

Key words: p-generalized Benney-Luke equation, Cauchy problem, Global existence.

1 Introduction

In this paper, we are interested in considering the Cauchy problem for the p-generalized Benney-Luke equation

$$\Phi_{tt}-\Delta\Phi+\sqrt{\varepsilon}(a\Delta^2\Phi-b\Delta\Phi_{tt})+\varepsilon(B\Delta^2\Phi_{tt}-A\Delta^3\Phi)+f(\Phi)=0, \tag{1.1}$$

$$\Phi(0,x)=\Phi_0(x), \qquad \Phi_t(0,x)=\Phi_1(x), \tag{1.2}$$

where $\Phi(x,t)$ denotes an unknown function of $x\in\mathbb{R}^n$ $(1\le n\le 10)$ and $t\in\mathbb{R}^+$, Δ is the Laplace operator, ∇ is the Hamilton operator, ε,a,b,A, and B are positive constants. The nonlinear term is defined as

$$f(\Phi)=\varepsilon f_p(\Phi)+\beta f_m(\Phi), \quad \beta\in\mathbb{R},$$

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where

$$f_p(\Phi)=\Phi_t\Delta_p\Phi+2\nabla^p\Phi\cdot\nabla\Phi_t,\quad f_m(\Phi)=\nabla\left(|\nabla\Phi|^m\nabla\Phi\right),$$

and ∇^p and Δ^p are denoted as

$$\nabla^p\Phi=((\partial_{x_1}\Phi)^p,...(\partial_{x_n}\Phi)^p),\tag{1.3}$$

and

$$\Delta_p\Phi=\nabla\cdot(\nabla^p\Phi)=\sum_{i=1}^n\partial_{x_i}(\partial_{x_i}\Phi)^p,\tag{1.4}$$

respectively. The constant m is positive and p satisfies

$$p\in\mathbb{N},\quad \text{or } p=\frac{m_1}{m_2}\geq 1,\quad m_1,\ m_2 \text{ are relative prime odd numbers,}\tag{1.5}$$

which guarantees the definition of Δ_p is reasonable.

The Benney-Luke equation,

$$\Phi_{tt}-\Delta\Phi+\mu(a\Delta^2\Phi-b\Delta\Phi_{tt})+\varepsilon(\Phi_t\Delta\Phi+2\nabla\Phi\cdot\nabla\Phi_t)=0,\tag{1.6}$$

is a model to describe the dispersive and the evolution of weakly nonlinear, long water waves of small amplitude, which was first derived by Benney and Luke when $a=\frac{1}{6}$ and $b=\frac{1}{2}$ with no surface tension ($\alpha=a-b+\frac{1}{3}=0$) (see [2]). Later, via the theory of weakly nonlinear long wave propagation in shallow water, Pego and Quintero [9] showed that the evolution of three-dimensional water waves with surface tension can be reduced to study the solution $\Phi(x,t)$ of the isotropic equation (1.6). Where Φ is the velocity potential on the domain, ε (nonlinearity coefficient) is the amplitude parameter and $\mu=(h_0/L)^2$ shows the long-wave parameter (dispersion coefficient). h_0 presents the equilibrium depth and L stands for horizontal length of motion. After rescaling the variables, we can suppose the constants a and b are positive and such that $a-b=\alpha-\frac{1}{3}\neq 0$, where α is the Bond number. A significant fact is that the Benney-Luke equation (1.6) can formally reduce to the Korteweg-de Vries equation and the Kadomtsev-Petviashvili equation or the Boussinesq equation in appropriate limits when we seek wave forms propagating predominantly in one direction slowly evolving in time and having weak transverse variation (see [7, 9]). The results about solitary waves, nonlinear stability and exact solutions for (1.6) were researched in [1, 5, 7–11, 14, 15, 17, 20].

Recently, Quintero [13] proved the existence and analyticity of the lump solutions (finite energy solitary waves) for the following generalized Benney-Luke equation

$$\Phi_{tt}-\Delta\Phi+\mu(a\Delta^2\Phi-b\Delta\Phi_{tt})+\varepsilon(\Phi_t\Delta_p\Phi+2\nabla^p\Phi\cdot\nabla\Phi_t)=0,\tag{1.7}$$

where ∇^p and Δ^p are denoted as (1.3) and (1.4) respectively. For the Cauchy problem of the equation (1.7), by means of the Strichartz estimates and the properties of the commutators of Kato-Ponce type, González [4] established the local and global well-posedness