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A Characterization of Weak n-Tilting Modules

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Abstract. It is well known that weak n-tilting modules are vital in tilting theory, which are generalizations of n-tilting and n-cotilting modules. The aim of this paper is to give a new characterization of weak n-tilting modules. In order to do that, we introduce the notion of weak n-star modules. We study more deeply the properties of them. Moreover, connections between (co) star and weak n-star modules are given.

AMS subject classifications: 16D90, 16E30, 16D40

Key words: Weak n-star module, Weak n-tilting module, n-Quasi-flat module.

1 Introduction and preliminaries

Throughout this paper, R is an associative ring with an identity. Denote by $\mathcal{F}$ the classes of flat left R-modules, respectively. We write R-Mod for the category of all left R-modules. For an R-module,we denote $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ as its character module. Moreover, $\mathrm{Add}(M)$ ($\mathrm{Prod}(M)$, respectively) denotes the class consisting of modules of all direct summands of arbitrary direct sums (products, respectively) of copies of M. We write $\mathrm{Gen}_n(T) := \{M \in R\text{-Mod} \mid \text{there exists an exact sequence } T_n \to \cdots \to T_1 \to M \to 0 \text{ with } T_i \in \mathrm{Add}\,T\}$ and $\mathrm{Cogen}_n(T) := \{M \in R\text{-Mod} \mid \text{there exists an exact sequence } 0 \to M \to T_1 \to \cdots \to T_n \text{ with } T_i \in \mathrm{Prod}\,T\}$. Note that there are clear inclusions between categories $\mathrm{Gen}_{n+1}(T) \subseteq \mathrm{Gen}_n(T)$ and $\mathrm{Cogen}_{n+1}(T) \subseteq \mathrm{Cogen}_n(T)$. We will say that an R-module T is reflexive under the functor $(-)^+ = \mathrm{Hom}_R(-,\mathbb{Q}/\mathbb{Z})$, if $T \cong T^{++}$.

$$^{\perp_{i\geq 1}}\mathcal{M} := \{N \in R\text{-Mod} \mid \mathrm{Ext}^i_R(N,M) = 0 \text{ for all } M \in \mathcal{M}, i \geq 1\}$$
$$\mathcal{M}^{\perp_{i\geq 1}} := \{N \in R\text{-Mod} \mid \mathrm{Ext}^i_R(M,N) = 0 \text{ for all } M \in \mathcal{M}, i \geq 1\}$$
$$^{\perp_{1\leq i\leq n}}\mathcal{M} := \{N \in R\text{-Mod} \mid \mathrm{Ext}^i_R(N,M) = 0 \text{ for all } M \in \mathcal{M}, 1 \leq i \leq n\}$$
$$\mathcal{M}^{\perp_{1\leq i\leq n}} := \{N \in R\text{-Mod} \mid \mathrm{Ext}^i_R(M,N) = 0 \text{ for all } M \in \mathcal{M}, 1 \leq i \leq n\}$$

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$$^{\top_{i\geq 1}}\mathcal{M} := \{N\in \text{Mod-}R\,|\,\text{Tor}_i^R(N,M)=0 \text{ for all } M\in\mathcal{M}, i\geq 1\}$$
$$^{\top_{1\leq i\leq n}}\mathcal{M} := \{N\in \text{Mod-}R\,|\,\text{Tor}_i^R(N,M)=0 \text{ for all } M\in\mathcal{M}, 1\leq i\leq n\}.$$

If $\mathcal{M}$ consists of a single object M, we write $^{\perp_{i\geq 1}}M$, $M^{\perp_{i\geq 1}}$, $^{\perp_{1\leq i\leq n}}M$, $M^{\perp_{1\leq i\leq n}}$, $^{\top_{i\geq 1}}M$, $^{\top_{1\leq i\leq n}}M$.

Tilting modules and cotilting modules, which generalize the progenerators and injective cogenerators, respectively, have drawn more and more academic interest in representation theory and homological algebra theory. The classical tilting modules were first considered by Brenner–Butler [1], Bongartz [2] and Happel and Ringel [3] etc. Colpi and Trlifaj [4] investigated (not necessarily finitely generated) tilting modules of projective dimension ≤ 1. Angeleri Hügel and Coelho [5], Bazzoni [6] did some work on tilting modules of finitely generated projective dimension $\leq n$ and extended them to the setting of (infinitely generated) modules over arbitrary rings. The well known Bazzoni's characterization of n-tilting modules (n-cotilting modules, respectively) says that an R-module M is n-cotilting (n-tilting, respectively) if and only if $\text{Cogen}_n(M) = {}^{\perp_{i\geq 1}}M$ ($\text{Gen}_n(M) = M^{\perp_{i\geq 1}}$, respectively). He [7] showed that the $^{\perp_{i\geq 1}}M$ ($M^{\perp_{i\geq 1}}$, respectively) can be replaced by $^{\perp_{1\leq i\leq n}}M$ ($M^{\perp_{1\leq i\leq n}}$, respectively). The infinitely generated tilting modules have been applied to various areas of module theory. For instance, the finitistic dimension conjectures are related to the tilting theory.

It is known that the character module of every tilting module is cotilting, but the converse is not true. Motivated by this, Yang, Yan, and Zhu [8] introduced the notion of weak n-tilting modules, which can be also viewed as a generalization of Tor-tilting modules in [9]. They defined a module W as a weak n-tilting module provided that (W1) $\text{fd}\,W\leq n$; (W2) $\text{Tor}_i^R((W^{(\kappa)})^+,W)=0$ for all $i\geq 1$ and any cardinal κ; and (W3) there is an exact sequence $0\to C_n\to C_{n-1}\to\cdots\to C_0\to R^+\to 0$, where $C_i\in\text{Prod}\,W^+$. And W is a partial weak n-tilting module if it satisfies $(W1)$ and $(W2)$. They proved that a module is weak n-tilting if and only if its character module is n-cotilting. Then the "triangular relation" was given, that is, all tilting modules are weak n-tilting modules, and M is a weak n-tilting module if and only if M^+ is an n-cotilting module.

Bazzoni and Šťovíček proved in [10] that all tilting modules are of finite type. In other words, any tilting class is determined by a class of modules of finite projective dimension admitting a projective resolution consisting of finitely generated projective modules. Yang, Yan, and Zhu showed that weak n-tilting modules of type FP_{n+1} are of finite type, and if W is a weak n-tilting module of type CP_{n+1}, then it is of finite type if and only if $W^{\perp_{i\geq 1}}$ is closed under direct sums. Furthermore, they gave that weak n-tilting modules of finite type are partial n-tilting. As an application of weak n-tilting modules, they characterized when all Gorenstein projective modules are Gorenstein flat. Based on their work, it is natural to consider the characterizations of weak-tilting modules. Yuan and Yao discussed some basic properties and gave equivalent descriptions of them in [16]. We want to characterize them in a new way by weak n-star modules.

$*$-modules were investigated by Menini and Orsatti [11] and Colpi [12]. Later on, $*^n$-modules (i.e., selfsmall n-star modules) were considered in [13]. They are also gen-