J. Math. Study
doi: 10.4208/jms.v57n2.24.03

Vol. **57**, No. 2, pp. 164-177
June 2024

# Boundedness and Compactness of Multilinear Singular Integrals on Morrey Spaces

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Received April 20, 2023; Accepted April 7, 2024;
Published online June 5, 2024.

**Abstract.** In this paper, we consider the boundedness and compactness of the multilinear singular integral operator on Morrey spaces, which is defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} R(A;x,y) f(y) dy,$$

where $R(A;x,y) = A(x) - A(y) - \nabla A(y) \cdot (x-y)$ with $D^\beta A \in BMO(\mathbb{R}^n)$ for all $|\beta| = 1$. We prove that $T_A$ is bounded and compact on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ for all $1 < p < \infty$ with $\Omega$ and $A$ satisfying some conditions. Moreover, the boundedness and compactness of the maximal multilinear singular integral operator $T_{A,*}$ on Morrey spaces are also given in this paper.

**AMS subject classifications**: 42B20, 42B25, 47G10
**Key words**: Multilinear operator, compactness, rough kernel, Morrey space.

## 1 Introduction

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ with the area measure $d\sigma$. $\Omega$ is the homogenous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, i.e.,

$$\Omega(\lambda x') = \Omega(x'), \qquad \text{for any } \lambda > 0 \text{ and } x' \in \mathbb{S}^{n-1}, \tag{1.1}$$

and satisfies the vanishing moment condition of order $m-1$, i.e.,

$$\int_{\mathbb{S}^{n-1}} \Omega(x') x'^\beta d\sigma(x') = 0 \qquad \text{for all } \beta \in \mathbb{Z}_+^n \text{ with } |\beta| = m-1. \tag{1.2}$$

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Here and in the sequel, $\beta=(\beta_1,\cdots,\beta_n)\in\mathbb{Z}_+^n$ is a multi-indices , $|\beta|=\sum_{j=1}^n\beta_j$ and $x^\beta=\prod_{i=1}^n x_i^{\beta_i}$ where $x\in\mathbb{R}^n$.

In this paper, we want to consider the following multilinear singular integral operator, $T_A^m(m\geq 2)$, which was first introduced by Cohen [12] in 1981 for $m=2$, and Cohen and Gosselin [13] in 1982 for $m>2$. The definition is as follows:

$$T_A^m f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)dy, \tag{1.3}$$

where $R_m(A;x,y)$ denotes the $m$-th order Taylor series remainder of the function $A$ at $x$ expanded about $y$, that is,

$$R_m(A;x,y)=A(x)-\sum_{|\beta|<m}\frac{1}{\beta!}D^\beta A(y)(x-y)^\beta,$$

with $\beta!=\beta_1!\cdots\beta_n!$, $D^\beta A(x)=\frac{\partial^{|\beta|}A}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}(x)$. Denote $T_A=T_A^2$ simply.

One can easily see that for $m=1$, the multilinear singular integral operator $T_A^m$ returns to the classical commutators of singular integrals, which was first introduced by Calderón [5] in 1965. Soon afterward, in 1967, Bajsanski and Coifman [1] considered the $L^p$-boundedness of the commutators of singular integrals. For $m\geq 2$, it is a non-trivial generalization.

For $m=2$, Cohen [12] studied the $L^p$-boundedness of the multilinear operator $T_A$. Then in 1986, Cohen and Gosselin [14] proved that $T_A^m$ is bounded on $L^p(\mathbb{R}^n)$ for $m\geq 2$ if $\Omega\in Lip(\mathbb{S}^{n-1})$ satisfies (1.1), (1.2) and $D^\beta A\in BMO(\mathbb{R}^n)$ (bounded mean oscillation functions) for all $|\beta|=m-1$ by using the method of the good-$\lambda$ inequality. In 1994, for $m=2$, Hofmann [18] improved the result of Cohen, and proved that $\Omega\in\cup_{s>1}L^s(\mathbb{S}^{n-1})$ is a sufficient condition such that $T_A$ is bounded on $L^p(\mathbb{R}^n)$ for all $p$ with $1<p<\infty$.

**Theorem 1.1** ([18]). *Suppose* $\Omega\in\bigcup_{s>1}L^s(\mathbb{S}^{n-1})$ *satisfies* (1.1) *and* (1.2). *Then for* $A$ *with* $\nabla A\in BMO$, $T_A$ *is bounded on* $L^p(\mathbb{R}^n)\,(1<p<\infty)$ *with the bound* $C\sum_{|\beta|=1}\|D^\beta A\|_*$.

It is well known that the compact operator is an important concept in analysis. The commutators of many important operators in harmonic analysis are all compact operators on some suitable $L^p$ spaces and Morrey spaces (see [25], [2], [20], [21], [26] and the recent works [3], [4], [6–11], [16], [17]). The $L^p$-compactness of the commutators of singular integrals $T_A^m(m=1)$ was obtained by Uchiyama [25] in 1978. In 2017, Ding and the first author of this paper [16] also studied the $L^p$-compactness of $T_A^m$ for $m\geq 2$.

Moreover, the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ (see the definition in Section 2) were first introduced by Morrey [24] in 1938 to study the local behavior of solutions of second-order elliptic partial differential equations. Later the Morrey spaces were found to have many important applications to the Navier-Stokes equations, the Schrödinger equations, the elliptic equations with discontinuous coefficients, the potential analysis, and the heat