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Nonlinear Mixed Lie Triple Derivations by Local Actions on Von Neumann Algebras

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Abstract. As a generalization of global mappings, we study a class of non-global mappings in this note. Let $\mathcal{A}\subseteq B(\mathcal{H})$ be a von Neumann algebra without abelian direct summands. We prove that if a map $\delta:\mathcal{A}\to\mathcal{A}$ satisfies $\delta([[A,B]_*,C])=[[\delta(A),B]_*,C]+[[A,\delta(B)]_*,C]+[[A,B]_*,\delta(C)]$ for any $A,B,C\in\mathcal{A}$ with $A^*B^*C=0$, then δ is an additive $*$-derivation. As applications, our results are applied to factor von Neumann algebras, standard operator algebras, prime $*$-algebras and so on.

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1 Introduction and preliminaries

Let $\mathcal{A}$ be a $*$-algebra over the complex number field $\mathbb{C}$. Recalled that $\delta:\mathcal{A}\to\mathcal{A}$ is called a derivation (without the additive assumption) if $\delta(AB)=\delta(A)B+A\delta(B)$ for all $A,B\in\mathcal{A}$. More generally, δ is called a Lie derivation (without the additive assumption) by $\delta([A,B])=[\delta(A),B]+[A,\delta(B)]$ and a skew Lie derivation (without the additive assumption) by $\delta([A,B]_*)=[\delta(A),B]_*+[A,\delta(B)]_*$, where $[A,B]_*=AB-BA^*$ is the skew Lie product of A and B. A map $\delta:\mathcal{A}\to\mathcal{A}$ (without the linearity assumption) is called a Lie triple derivation if

$$\delta([[A,B],C])=[[\delta(A),B],C]+[[A,\delta(B)],C]+[[A,B],\delta(C)]$$

for any $A,B,C\in\mathcal{A}$. In recent years, Lie triple derivations have been studied intensively, see for example [2, 4, 8, 10, 14]. Very recently, many authors have paid attention to non-global nonlinear Lie triple derivations. Let $F:\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ be a map and $\mathcal{Q}$ be a proper subset of $\mathcal{A}$. $\delta:\mathcal{A}\to\mathcal{A}$ is called a non-global nonlinear Lie triple derivation if

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δ satisfies $\delta([[A,B],C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] + [[A,B],\delta(C)]$ for any $A,B,C \in \mathcal{A}$ with $F(A,B,C) \in \mathcal{Q}$. For more literature on this topic, please see [1,6,7,9,12,13].

Motivated by the above-mentioned work, we consider studying nonlinear non-global mixed Lie triple derivations. First, we give the definition of mixed Lie triple derivations. A map $\delta : \mathcal{A} \to \mathcal{A}$ (without the linearity assumption) is called a nonlinear mixed Lie triple derivation if

$$\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)]$$

for all $A,B,C \in \mathcal{A}$. Liang and Zhang [5] proved that every nonlinear mixed Lie triple derivation on factor von Neumann algebras is an additive $*$-derivation. Zhou, Yang, and Zhang [15] generalized the above result to the case of prime $*$-algebras. Similarly, we can naturally define non-global mixed Lie triple derivations. Let $F : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a map and $\mathcal{Q}$ be a proper subset of $\mathcal{A}$. $\delta : \mathcal{A} \to \mathcal{A}$ is called a non-global nonlinear mixed Lie triple derivation if δ satisfies

$$\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)],C] + [[A,B]_*,\delta(C)]$$

for any $A,B,C \in \mathcal{A}$ with $F(A,B,C) \in \mathcal{Q}$.

In fact, non-global mappings are a generalization of global mappings. Therefore we need more new techniques and methods to solve the following problem. In this paper, we will concentrate on giving the structure of the nonlinear mixed Lie triple derivations on von Neumann algebras with $A^*B^*C=0$. Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a von Neumann algebra without abelian direct summands. We prove that if a map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies $\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)]$ for any $A,B,C \in \mathcal{A}$ with $A^*B^*C=0$, then δ is an additive $*$-derivation. As applications, our results are applied to factor von Neumann algebras, standard operator algebras, prime $*$-algebras, and so on.

Therefore, we need some notations and preliminaries about von Neumann algebras. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity I. $\mathcal{Z}_{\mathcal{A}} = \{Z \in \mathcal{A} : ZA = AZ\}$ for all $A \in \mathcal{A}$ is called the center of $\mathcal{A}$. From a ring theoretic perspective, standard operator algebras and factor von Neumann algebras are both primes, whereas von Neumann algebras are usually semiprime. Recall that an algebra $\mathcal{A}$ is prime if $A\mathcal{A}B = 0$ implies either $A=0$ or $B=0$. An algebra is a semiprime if $A\mathcal{A}A=0$ implies $A=0$. A projection P is called a central abelian projection if $P \in \mathcal{Z}_{\mathcal{A}}$ and $P\mathcal{A}P$ is abelian. We denote $\overline{A}$ as the central carrier of A, which is the smallest central projection satisfying $PA = A$. It is straightforward to check that the central carrier of $\mathcal{A}$ is the projection onto the closed subspace spanned by $\{MA(x) : M \in \mathcal{A}, x \in \mathcal{H}\}$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is sup $\{S \in \mathcal{Z}_{\mathcal{A}} : S = S^*, S \leq A\}$. If P is a projection, it is clear that $\underline{P}$ is the largest central projection $\leq P$. A projection P is said to be core-free if $\underline{P}=0$. It is not difficult to see that $\underline{P}=0$ if and only if $\overline{I-P}=I$.

Let $\mathcal{A}$ be a von Neumann algebra without abelian direct summands, then there exists a core-free projection, denoted by P_1, that is $\overline{P_1}=I$ and $\underline{P_1}=0$. Clearly, $P_1 \neq 0, I$. Throughout