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Blow-Up of Solutions to Boundary Value Problems of Coupled Wave Equations with Damping Terms on Exterior Domain

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Abstract. This paper is concerned with blow-up dynamics of solutions to coupled systems of damped inhomogeneous wave equations with power nonlinearities related to weight function $t^{\alpha}|x|^{\beta}$ and variable boundary conditions on an exterior domain. The damping terms investigated in this work contain weak damping terms and convection terms. In terms of the Neumann-type boundary conditions and Dirichlet-type boundary conditions, the non-existence of global solutions to the problems is demonstrated by constructing appropriate test functions and applying contradiction arguments, respectively. Our main new contributions are that the effects of damping terms and nonlinear terms on behaviors of solutions to the coupled inhomogeneous wave equations are analyzed. As far as the authors know, the results in Theorems 1.1-1.4 are new.

AMS subject classifications: 35L70, 58J45

Key words: Coupled system, inhomogeneous wave equations, exterior domain, test function technique, blow-up.

1 Introduction

In this paper, we mainly work on the boundary value problems for coupled systems of inhomogeneous wave equations with power nonlinearities depending on space and time variables as well as different boundary conditions. Namely, we consider the problem

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with Neumann-type boundary conditions

$$\begin{cases} u_{tt}-\Delta u+u_t-k\dfrac{x}{|x|^2}\cdot\nabla u=t^{\alpha}|x|^{\beta}|v|^p+w(x), & t>0,\ x\in\Omega^c,\\ v_{tt}-\Delta v+v_t-k\dfrac{x}{|x|^2}\cdot\nabla v=t^{\alpha}|x|^{\beta}|u|^q+w(x), & t>0,\ x\in\Omega^c,\\ (\dfrac{\partial u}{\partial \nu},\dfrac{\partial v}{\partial \nu})(t,x)=(f,g)(x), & t>0,\ x\in\partial\Omega \end{cases} \tag{1.1}$$

and the problem with Dirichlet-type boundary conditions

$$\begin{cases} u_{tt}-\Delta u+u_t-k\dfrac{x}{|x|^2}\cdot\nabla u=t^{\alpha}|x|^{\beta}|v|^p+w(x), & t>0,\ x\in\Omega^c,\\ v_{tt}-\Delta v+v_t-k\dfrac{x}{|x|^2}\cdot\nabla v=t^{\alpha}|x|^{\beta}|u|^q+w(x), & t>0,\ x\in\Omega^c,\\ (u,v)(t,x)=(f,g)(x), & t>0,\ x\in\partial\Omega. \end{cases} \tag{1.2}$$

Here, $\Delta=\sum_{i=1}^N\frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator, $-k\frac{x}{|x|^2}\cdot\nabla u$ and $-k\frac{x}{|x|^2}\cdot\nabla v$ stand for convection terms, where $k\in\mathbb{R}$. The dot $\cdot$ denotes inner product in $\mathbb{R}^N\,(N\geq 2)$. The exponents in nonlinear terms satisfy $-1\leq\alpha<\infty$, $\beta\in\mathbb{R}$, $1<p,q<\infty$. $\Omega=\{x\in\mathbb{R}^N\,\big|\,|x|\leq 1\}$ is the unit closed ball. $\Omega^c=\mathbb{R}^N\backslash\Omega$ represents the exterior domain of unit ball. ν represents the outward unit normal vector on $\partial\Omega$ relative to Ω^c. $f(x),g(x)\in L^1(\partial\Omega)$ are non-negative smooth functions.

Over the recent decades, the Cauchy problem of classical wave equation

$$\begin{cases} u_{tt}-\Delta u=|u|^p, & t>0,\ x\in\mathbb{R}^N,\\ (u,u_t)(0,x)=\varepsilon(u_0,u_1)(x), & x\in\mathbb{R}^N \end{cases} \tag{1.3}$$

has been investigated extensively (see detailed instructions in [9, 11, 20, 22, 23, 27, 28, 30, 33, 36, 37, 42, 43]). It is worthwhile to mention that problem (1.3) possesses the Strauss exponent $p_S(N)$. If $N=1$, it holds that $p_S(N)=\infty$. If $N\geq 2$, we recognize that $p_S(N)$ is the positive root of quadratic equation

$$-(N-1)p^2+(N+1)p+2=0.$$

The critical exponent means the threshold between the blow-up dynamics of the solution and the existence of a global (in time) solution with small initial values. John [20] proves that the critical exponent of problem (1.3) is $p_S(3)=1+\sqrt{2}$ in three space dimensions. Blow-up result of solution to problem (1.3) is illustrated for $1<p<p_S(3)$, while there exists global solution for $p>p_S(3)$. Strauss [37] conjectures that the solution to problem (1.3) blows up in finite time when $1<p<p_S(N)$, while the problem admits global solution when $p>p_S(N)$. Zhou and Han [45] establish an upper bound lifespan estimate of the solution to problem (1.3) in the variant coefficient case when $1<p<p_S(N)\,(N\geq 3)$ by