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Formation of Singularity for Compressible Euler Equations Outside a Ball in 3-D

Mengxuan Li[1] and Jinbo Geng*

School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China.

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Abstract. The initial boundary value problem for a compressible Euler system outside a ball in $\mathbf{R}^3$ is considered in this paper. Assuming the initial data have small and compact supported perturbations near a constant state, we show that the solution will blow up in a finite time, and the lifespan estimate can be estimated by the small parameter of the initial perturbations. To this end, a "tricky" test function admitting good behavior is introduced.

AMS subject classifications: 35Q31, 35L65, 76N15

Key words: Compressible Euler equations, exterior domain, blow-up, impermeable boundary condition.

1 Introduction

This paper is concerned about the following initial boundary value problem of compressible Euler system in an exterior domain

$$\begin{cases} \rho_t+\nabla\cdot(\rho u)=0, & (t,x)\in\mathbf{R}^+\times B_1^c, \\ (\rho u)_t+\nabla\cdot(\rho u\otimes u)+\nabla P=0, & (t,x)\in\mathbf{R}^+\times B_1^c, \\ u(0,x)=\varepsilon u_0(x),\ \ \rho(0,x)=\overline{\rho}+\varepsilon\rho_0(x), & x\in B_1^c, \\ u\cdot\nu\Big|_{\partial B_1^c}=0, \end{cases} \tag{1.1}$$

where B_1^c denotes the exterior domain outside a unit ball $B_1\subset\mathbf{R}^3$, which is centered at the origin, and ν denotes the unit outward normal of B_1^c to the boundary. Here $\rho=\rho(t,x)$ and $u=u(t,x)$ are real-valued unknown functions, representing the density and velocity of

*Corresponding author. *Email addresses:* `limengxuan@zjnu.edu.cn` (Li M), `jinbogeng@zjnu.cn` (Geng J)

the flow respectively, and $\overline{\rho}>0$ denotes a constant state for the density. We consider the state equation for the gas in the form

$$P=A\rho^{\gamma},$$

with $A>0,\gamma>1$ are constants. Also, $\varepsilon>0$ is a parameter representing the smallness of the initial perturbation of density and velocity satisfying

$$(u_0(x),\rho_0(x))\in C_0^{\infty}(B_1^c), \qquad supp(u_0(x),\rho_0(x))\subset\{x||x|\le R\}, \tag{1.2}$$

where $R>1$ is a fixed constant (in the following we may assume $R=2$ for convenience). The boundary condition $(1.1)_4$ corresponds to the impermeable boundary which is stationary and different from the free surface. It can be used to describe a simple type of boundary of an impermeable wall, such as the side of a wave tank or the hull of a ship.

There is extensive literature on the formation of singularity for compressible Euler equations. For the most relevant corresponding Cauchy problem with small perturbations, we refer to Sideris [20] in $\mathbf{R}^3$, Rammaha [18] in $\mathbf{R}^2$ and Jin-Zhou [8] in $\mathbf{R}^n(n=1,2,3)$. Also, the lifespan estimate is studied in [1,21,22].

Secchi [19] considered the exterior problem of the Euler equations for a barotropic inviscid compressible fluid in $\Omega\subset\mathbf{R}^2$, assuming the boundary $\partial\Omega$ is smooth and convex. The lifespan estimate from below (the largest time interval over which there exists a classical solution) was established. For some recent long-time behavior results on the related fluid models in the exterior domain, see [2], [16] and references therein.

Recently, the finite time blow-up result for compressible Euler system in the exterior domain in $\mathbf{R}^n(n=2,3)$ are established in [4], where the obstacle is convex and abstract test functions are introduced. In this paper, we consider the finite blow-up outside a ball in $\mathbf{R}^3$, in which case an explicit test function can be found, and hence the asymptotic behavior of which is easy to get. Such kind of method can be found in [8,13,14,27].

Remark 1.1. We should mention that there are so many other long-term behavior results including singularity formation and global existence for compressible Euler and Navier-Stokes systems that it is impossible for us to list all of them, see [5,7,9,11,15,23–25] and references therein.

Our main result states as follows.

Theorem 1.1. *Assuming the initial perturbations $u_0(x),\rho_0(x)$ satisfy* (1.2) *and*

$$\varepsilon\left(\int_{B_1^c}\rho_0(x)\phi(x)dx+\int_{B_1^c}u_0(x)\cdot\nabla\phi dx\right)\triangleq\frac{1}{2}C\varepsilon>0,$$

where $\phi(x)$ is defined in (3.1)*. We also assume the initial velocity $u_0(x)$ satisfies the necessary compatibility conditions to some order, i.e.*

$$\partial_t^k u_0(x)\cdot\nu=0, \quad x\in\partial B_1^c,\ k=0,1.$$