J. Math. Study
doi: 10.4208/jms.v57n2.24.07

Vol. **57**, No. 2, pp. 230-242
June 2024

On Smooth Families of Diffeomorphic Manifolds and Their Distribution Function

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Received May 16, 2023; Accepted June 30, 2023;
Published online June 5, 2024.

Abstract. We present a formula for the third derivative of the distribution function of a regular function on a domain of $\mathbb{R}^{n+1}$, and a further discussion of the extra assumption that the function is harmonic. The present work builds on [3,5].

AMS subject classifications: 53-XX

Key words: Level set, third variation.

1 Introduction

Given a regular function u on a domain of $\mathbb{R}^{n+1}$, its distribution function $V(t)$ is the Hausdorff measure of its level subset t. The expressions and some inequalities about the first and the second derivative of V were studied in [3,5]. In the present note, we present a formula for V'''; namely

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set, and that $u \in C^2(\overline{\Omega})$ is constant on each connected component of $\partial\Omega$. Let t_1, t_2 be such that $-\infty \leq t_1 < t_2 \leq \infty$ and suppose that in $\{x \in \Omega | t_1 \leq u(x) \leq t_2\}$ there exists a positive constant c such that $|\nabla u| \geq c$. Then, for any $t \in (t_1, t_2)$ we have that*

$$V'''(t) = \int_{\{u=t\}} \frac{6S_1^2}{|\nabla u|^3} dH^n + \int_{\{u=t\}} \frac{2S_2}{|\nabla u|^3} dH^n + \int_{\{u=t\}} \frac{-9S_1 \Delta u}{|\nabla u|^4} dH^n + \int_{\{u=t\}} \frac{3(\Delta u)^2}{|\nabla u|^5} dH^n + \int_{\{u=t\}} \frac{u_{NNN}}{|\nabla u|^4} dH^n, \tag{1.1}$$

where H^n is the n-dimensional Hausdorff measure. Here above S_r, for each $r \in \{0, \ldots, n\}$ stands for the r-th elementary symmetric functions of the principal curvatures of $\{u=t\}$; moreover N is the normal unit vector to $\{u=t\}$, and u_{NNN} is the third derivative of u in the N-direction.

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Remark 1.1. The assumptions in Theorem 1.1 on the non-vanishing of the gradient of u are needed to build on [3]. In order to apply [5] it is needed that the level sets of u in $\{x\in\Omega|t_1\le u(x)\le t_2\}$ are mutually diffeomorphic: this is guaranteed by the following observation by Professor LeBrun. Let T be the value of u on the boundary $\partial\Omega$, and assume for a moment that T does not belong to $[t_1,t_2]$. Since the closure of the domain is compact, the restriction of the function u to $u^{-1}([t_1,t_2])$ is proper. But we have assumed that the gradient of u is everywhere non-zero, so now we are in a position to apply Theorem 3.1 of [4] which ensures that the level sets of u are in $\{x\in\Omega|t_1\le u(x)\le t_2\}$ diffeomorphic. The above discussion assumes that T does not belong to $[t_1,t_2]$. But with assumptions of Theorem 1.1 the only alternatives left are that either $T=t_1$ and the gradient of u is inward-pointing, or $T=t_2$ and the gradient of u is outward-pointing; in both cases, the level set where $u=T$ is a smooth compact hypersurface, still diffeomorphic to the other level sets.

In particular, all the level sets of u in $\{x\in\Omega|t_1\le u(x)\le t_2\}$ have the same number of connected components, and we can then apply, to the sub-domains represented by each connected component, the argument that we will describe for one connected component. Incidentally, each component of a level set is now a smooth connected compact real hypersurface in $\mathbb{R}^{n+1}$, and so disconnects $\mathbb{R}^{n+1}$ into an ¡°inside¡± and an ¡°outside¡± by the smooth Jordan-Brouwer separation theorem. So we could then just restrict to the regions inside appropriately chosen components of the level surfaces.

The note is organized as follows: the second section contains a recap of the results needed from [3, 5]; the third section contains the proof of Theorem 1.1, and the third section contains a discussion on the case when the extra assumption of harmonicity of u is assumed.

2 Background environment

In the work of Reilly [5] it is considered family, depending smoothly from the parameter $t\in(-\epsilon,\epsilon)$, of smooth immersions

$$x=X_t:M^n\to S^{n+1}(c),$$

where M^n is a smooth manifold of real dimension n without boundary, and $S^{n+1}(c)$ is the simply connected $n+1$ dimensional space form of curvature c. Whence M^n, as smooth hypersurface of $S^{n+1}(c)$, has principal curvatures $k_1,\dots,k_n$. As S_r, for each $r\in\{0,\dots,n\}$ is labeled the r-th elementary symmetric functions of the principal curvatures; hence, for example, $S_0=1$, $S_1=k_1+\dots+k_n$, $\dots$, $S_n=k_1\cdot\dots\cdot k_n$. To recall the main formula in [5], we consider the case when $c=0$, letting $<,>$ be the Euclidean metric on $\mathbb{R}^{n+1}$ and setting $P:=<X,N>$ be the support function, where N is the normal vector to M^n. Moreover, we set $Q:=\frac{1}{2}|X|^2$, $\xi:=\frac{\partial X}{\partial t}$, $\lambda:=<N,\frac{\partial X}{\partial t}>$; the volume form is denoted as $d\mu$. Finally, we let B be is the shape operator, i.e., B is the symmetric linear transformation associated