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Spectral Element Methods for Stochastic Differential Equations with Additive Noise

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Abstract. In this paper, we propose numerical schemes for stochastic differential equations driven by white noise and colored noise, respectively. For this purpose, we first discretize the white noise and colored noise, and give their regularity estimates. Then we use spectral element methods to solve the corresponding stochastic differential equations numerically. The approximation errors are derived, and the numerical results demonstrate high accuracy of the proposed schemes.

AMS subject classifications: 65M70, 65L60, 41A10, 60H35

Key words: Stochastic differential equations, spectral element methods, high accuracy

1 Introduction

Many problems in science and engineering can be characterized by differential equations. However, deterministic differential equations can not reveal the essence of these problems well due to the existence of accidental phenomena and small probability events. Therefore, many scholars consider stochastic differential equations to describe these problems, see [10,22,23] and etc, and the references therein.

Since the solution of a stochastic differential equation (SDE) is a stochastic process, it is difficult to reveal visually the information the process contains. Recently, there are many works to study the numerical solution for stochastic ordinary differential equations (SDEs). Euler-Maruyama method, Milstein method and Runge-Kutta method were proposed to solve SDEs numerically, see [3,4,13,16,17,19,20,31] and the references therein.

On the other hand, many authors investigated the numerical methods for stochastic partial differential equations (SPDEs). Allen *et al.* [1], Mcdonald [21], Gyöngy [14, 15] used finite difference method to study the numerical solutions of linear SPDEs driven by additive white noise. Also, Walsh [28], Du and Zhang [11] studied finite element method

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for linear SPDEs driven by special noises. Moreover, Cao *et al.* [6, 7, 8], Theting [26, 27], Yan [29] and etc, have also made some contributions.

As we know, spectral methods take orthogonal polynomials (such as Legendre, Chebyshev, Jacobi, Laguerre and Hermite polynomials) as the basis functions to approximate the solutions of problems in mathematical physics, and tend to have higher accuracy, see [5, 12, 25]. Recently, some scholars are trying to study SPDEs by spectral methods. Shardlow [24] took the eigenvectors of $-\frac{\partial^2}{\partial x^2}$ subject to Dirichlet boundary condition as the basis functions and used spectral method to approximate the noise. Cao and Yin [9] studied spectral Galerkin method for stochastic wave equations driven by space-time white noise. In addition, Yin and Gan [30] proposed the Chebyshev spectral collocation method to solve a certain type of stochastic delay differential equations.

In the present work, we will try to solve stochastic equations by using spectral element methods. We begin with taking the SDEs driven by white noise into consideration. As we know, the regularity of the solutions of the original stochastic differential equations plays an important role in priori error estimates of the numerical solutions. Unfortunately, the regularity estimates are usually very weak because of the existence of white noise. In order to overcome this difficulty, we will approximate the white noise process by piecewise constant random process as in [1], and apply the Legendre spectral element method for the corresponding stochastic differential equation. We prove the numerical solution converges to the original solution. Moreover, the accuracy of the proposed scheme is indicated by the provided computational results. The other part is that we use spectral element methods to solve stochastic differential equations driven by colored noise numerically. Analogous to the way to deal with white noise, it is nature to use an finite dimensional noise to discretize the colored noise (see [11]), which could improve the regularity of the solutions of the original stochastic differential equations. Therefore, the relevant stochastic differential equation is capable of being approximated by the Legendre spectral element scheme. Furthermore, the numerical results show the high accuracy of spectral element methods.

This paper is organized as follows. In the next section, we study numerical solutions of stochastic differential equations driven by white noise using spectral element methods, and the error estimates as well as the numerical results are presented. In Section 3, we use spectral element methods to solve stochastic differential equations driven by colored noise numerically. The final section is for conclusion remarks.

2 Numerical solutions of SDEs driven by white noise

We consider the following stochastic problem driven by white noise

$$\begin{cases} -u''(x)+bu(x)=g(x)+\dot{W}(x), & x\in I:=(0,1), \\ u(0)=u(1)=0, \end{cases} \tag{2.1}$$

where $\dot{W}(x)$ denotes white noise, $g(x)$ is a deterministic function and b is a constant.