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Convergence Analysis of an Unconditionally Energy Stable Linear Crank-Nicolson Scheme for the Cahn-Hilliard Equation

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Abstract. Efficient and unconditionally stable high order time marching schemes are very important but not easy to construct for nonlinear phase dynamics. In this paper, we propose and analysis an efficient stabilized linear Crank-Nicolson scheme for the Cahn-Hilliard equation with provable unconditional stability. In this scheme the nonlinear bulk force are treated explicitly with two second-order linear stabilization terms. The semi-discretized equation is a linear elliptic system with constant coefficients, thus robust and efficient solution procedures are guaranteed. Rigorous error analysis show that, when the time step-size is small enough, the scheme is second order accurate in time with a prefactor controlled by some lower degree polynomial of $1/\varepsilon$. Here ε is the interface thickness parameter. Numerical results are presented to verify the accuracy and efficiency of the scheme.

AMS subject classifications: 65M12, 65M15, 65P40

Key words: phase field model, Cahn-Hilliard equation, unconditionally stable, stabilized semi-implicit scheme, high order time marching.

1 Introduction

In this paper, we consider numerical approximation for the Cahn-Hilliard equation

$$\begin{cases} \phi_t = -\gamma\Delta(\varepsilon\Delta\phi - \frac{1}{\varepsilon}f(\phi)), & (x,t)\in\Omega\times(0,T], \\ \phi|_{t=0} = \phi_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

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with Neumann boundary condition

$$\partial_n\phi=0, \quad \partial_n(\varepsilon\Delta\phi-\frac{1}{\varepsilon}f(\phi))=0, \quad x\in\partial\Omega. \tag{1.2}$$

Here $\Omega\in R^d, d=2,3$ is a bounded domain with a locally Lipschitz boundary, n is the outward normal, T is a given time, $\phi(x,t)$ is the phase-field variable. Function $f(\phi)=F'(\phi)$, with $F(\phi)$ is a given energy potential with two local minima, e.g. the double well potential

$$F(\phi)=\frac{1}{4}(\phi^2-1)^2.$$

The two minima of F produces two phases, with the typical thickness of the interface between two phases given by ε. γ is a time relaxation parameter, its value is related to the time unit used in a physical process.

Eq. (1.1) is a fourth-order partial differential equation, which is not easy to solve using a finite element method. However, if we introduce a new variable μ, called chemical potential, for $-\varepsilon\Delta\phi+\frac{1}{\varepsilon}f(\phi)$, Eq. (1.1) can be rewritten as a system of two second order equations

$$\begin{cases}\phi_t=\gamma\Delta\mu, & (x,t)\in\Omega\times(0,T],\\ \mu=-\varepsilon\Delta\phi+\frac{1}{\varepsilon}f(\phi), & (x,t)\in\Omega\times(0,T],\\ \phi|_{t=0}=\phi_0(x), & x\in\Omega.\end{cases} \tag{1.3}$$

The corresponding Neumann boundary condition reads

$$\partial_n\phi=0, \quad \partial_n\mu=0, \quad x\in\partial\Omega. \tag{1.4}$$

The Cahn-Hilliard equation was originally introduced by Cahn-Hilliard [6] to describe the phase separation and coarsening phenomena in non-uniform systems such as alloys, glasses and polymer mixtures. If the term $\Delta\mu$ in equation (1.3) is replaced with $-\mu$, one get the Allen-Cahn equation, which was introduced by Allen and Cahn [2] to describe the motion of anti-phase boundaries in crystalline solids. The Cahn-Hilliard equation and the Allen-Cahn equation are two widely used phase-field model. In a phase-field model, the information of interface is encoded in a smooth phase function ϕ. In most parts of the domain Ω, the value of ϕ is close to local minima of F. The interface is a thin layer of thickness ε connecting regions of different local minima. It is easy to deal with dynamical process involving morphology changes of interfaces using phase-field models. For this reason, phase field models have been the subject of many theoretical and numerical investigations (cf., for instance, [7–9,12,14,15,17,19,22,23,30,35]).

However, numerically solving the phase-field equations is not an easy task, since the small parameter ε in the Cahn-Hilliard equation makes the equation very stiff and requires a high spatial and temporal grid resolution. To design an energy stable scheme,