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# A Diagonalized Legendre Rational Spectral Method for Problems on the Whole Line

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**Abstract.** A diagonalized Legendre rational spectral method for solving second and fourth order differential equations are proposed. Some Fourier-like Sobolev orthogonal basis functions are constructed which lead to the diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series. Numerical results demonstrate the effectiveness of this approach.

**AMS subject classifications**: 65N35, 41A20, 33C45, 35J25, 35J40, 35K05

**Key words**: Legendre rational spectral method, Sobolev orthogonal functions, elliptic boundary value problems, heat equation, numerical results.

## 1 Introduction

Many science and engineering problems are set in unbounded domains, such as fluid flows in an infinite strip, nonlinear wave equations in quantum mechanics and so on. How to accurately and efficiently solve such problems is a very important and difficult subject, since the unboundedness causes considerable theoretical and practical challenges. There are several ways for their numerical simulations. Usually we restrict calculations to some bounded subdomains and impose certain artificial boundary conditions. It is easy to be performed, but it lowers the accuracy sometimes. The second way is to use spectral method associated with some orthogonal systems on the unbounded domains, such as the Laguerre and Hermite spectral method [1, 2, 5, 6, 8–10, 13, 16–19]. However, since the Laguerre/Hermite Gauss points are too concentrated near zero, the approximation results are usually not ideal, especially where the points are far away from zero. The third method is to change original problems by variable transformations to certain singular problems on finite intervals, and then use Jacobi approximation to resolve the

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resulting problems [7]. The fourth effective method is to use algebraically mapped Legendre, Chebyshev or Jacobi functions to approximate the differential equations, i.e., the so-called Legendre, Chebyshev or Jacobi rational spectral method [3,4,11,12,21,22]. Compared with the first three methods, we prefer the last way, since the distribution of the Gauss points is more reasonable than that of Laguerre and Hermite Gauss points.

As is well known, the utilization of Legendre rational functions usually leads to a highly sparse algebraic system (a nine-diagonal matrix for second order problems and a seventeen-diagonal matrix for fourth order problems), the condition numbers increase as $\mathcal{O}(N^2)$ for the second order problem and $\mathcal{O}(N^4)$ for the fourth order problem. However, in many cases, researchers still want a set of Fourier-like basis functions for a diagonalized algebraic system [14,15,20]. Motivated by [14,15,20], the main purpose of this paper is to construct the Fourier-like Sobolev orthogonal basis functions and propose the diagonalized Legendre rational spectral method for second and fourth problems on the whole line.

The main advantages of the suggested algorithm include: (i) The exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series, respectively; (ii) The condition numbers for the resulting algebraic systems are equal to 1; (iii) The computational cost is much less than that of the classical Legendre rational spectral method.

This paper is organized as follows. In Section 2, we introduce the modified Legendre rational functions and its basic properties. In Section 3, we construct the Sobolev orthogonal Legendre rational functions corresponding to the second order elliptic equation, the fourth order elliptic equation and the nonlinear heat equation, and propose the diagonalized Legendre rational spectral methods. Some numerical results are presented in Section 4 to demonstrate the effectiveness and accuracy.

## 2 Modified legendre rational functions

We first recall the Legendre polynomials. Let $I=\{y \mid -1<y<1\}$ and $L_k(y)$ be the Legendre polynomial of degree $k$, which is the eigenfunction of the singular Sturm-Liouville problem:

$$\partial_y((1-y^2)\partial_y L_k(y))+k(k+1)L_k(y)=0, \quad k\geq 0. \tag{2.1}$$

The set of all Legendre polynomials forms a complete $L^2(I)$-orthogonal system, namely,

$$\int_I L_k(y)L_l(y)dy=\frac{2}{2k+1}\delta_{k,l}, \tag{2.2}$$

where $\delta_{k,l}$ is the Kronecker function. By virtue of (2.1) and (2.2), we have

$$\int_I \partial_y L_k(y)\partial_y L_l(y)(1-y^2)dy=\frac{2k(k+1)}{2k+1}\delta_{k,l}. \tag{2.3}$$