

Chebyshev Spectral Method for Volterra Integral Equation with Multiple Delays

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Received October 9, 2017; Accepted May 25, 2018

Abstract. Numerical analysis is carried out for the Volterra integral equation with multiple delays in this article. Firstly, we make two variable transformations. Then we use the Gauss quadrature formula to get the approximate solutions. And then with the Chebyshev spectral method, the Gronwall inequality and some relevant lemmas, a rigorous analysis is provided. The conclusion is that the numerical error decay exponentially in L^∞ space and $L^2_{\omega^c}$ space. Finally, numerical examples are given to show the feasibility and effectiveness of the Chebyshev spectral method.

AMS subject classifications: 65R20, 45E05

Key words: Volterra integral equation, multiple delays, Chebyshev spectral method, Gronwall inequality, convergence analysis.

1 Introduction

In this paper, we consider the Volterra integral equation with multiple delays of the form

$$y(t) - \sum_{l=1}^M \int_0^t k_l(t, \xi) y(a_l \xi) d\xi = f(t), \quad (1.1)$$

where the unknown function $y(t)$ is defined on $0 \leq t \leq T < +\infty$. The source function $f(t)$ and kernel function $k_l(t, \xi)$ ($l=1, 2, \dots, M$) are given sufficiently smooth functions, with the condition that M is a given natural number, $0 < a_l \leq 1$.

These kinds of equations arise in many areas, such as the Mechanical problems of physics, the movement of celestial bodies problems of astronomy and the problem of biological population original state changes. It is also applied to network reservoir, storage

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system, material accumulation, etc., and solve a lot problems from mathematical models of population statistics, viscoelastic materials and insurance abstracted. Due to the significance of these equations which have played in many disciplines, they must be solved efficiently with proper numerical approach. In recent years, these equations have been extensively researched, such as collocation methods [3–5,21], Taylor series methods [10], linear multistep methods [14], spectral analysis [1,2,7–9,11,18–20]. In fact, spectral methods have excellent error properties called "exponential convergence" which is the fastest possible. There are many also many spectral methods to solve Volterra integral equations, for example Legendre spectral-collocation method [18], Jacobi spectral-collocation method [8], spectral Galerkin method [20], Chebyshev spectral method [11] and so on. As the Chebyshev points are easier to be obtained than in [2], in this paper we are going to use the Chebyshev spectral method to deal with the Volterra integral equation with multiple delays. Meanwhile the error estimate of the $L^2_{\omega^c}$ norm is observed in our article, while in I. Ali, H. Brunner and T. Tang's is not. The third difference is that I. Ali, H. Brunner and T. Tang's article only has two delay terms, while our article has M delay terms. Compared to the work by Zhang Ran in [22], the novelty is that the delay term in their article is in the integral term while in ours the delay terms are in the integrand functions. In a word, we provide rigorous error analysis by Chebyshev spectral method for the Volterra integral equation with multiple delays that theoretically justifies the spectral rate of convergence in this paper. Numerical tests are also presented to verify the theoretical result.

We organize this paper as follows. In Section 2, we introduce the Chebyshev spectral method. Some knowledge which is important for the derivation of the mail result is given in the next section. We carry out the convergence analysis in Section 4 and Section 5 contains numerical tests which are illustrated to confirm the theoretical result. In the end a conclusion is given in Section 6.

Throughout the paper C denotes a positive constant that is independent of N , but depends on other given conditions.

2 Chebyshev spectral method

In this section, we review Chebyshev spectral method. Firstly we use the variable transformations as follow

$$t = \frac{T(1+x)}{2}, \quad \zeta = \frac{T(1+s)}{2},$$

and if we note that

$$u(x) = y\left(\frac{T(1+x)}{2}\right), \quad \hat{k}_l(x,s) = \frac{T}{2}k_l\left(\frac{T(1+x)}{2}, \frac{T(1+s)}{2}\right), \quad g(x) = f\left(\frac{T(1+x)}{2}\right),$$

then (1.1) can be written as

$$u(x) - \sum_{l=1}^M \int_{-1}^x \hat{k}_l(x,s)u(a_l s + a_l - 1)ds = g(x), \quad x \in [-1,1]. \tag{2.1}$$

Set the collocation points as the set of $N+1$ Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points $\{x_i\}_{i=1}^N$, (see, e.g., [6]). Assume that equation (2.1) hold at x_i

$$u(x_i) - \sum_{l=1}^M \int_{-1}^{x_i} \hat{k}_l(x_i, s) u(a_l s + a_l - 1) ds = g(x_i). \quad (2.2)$$

The main difficulty in obtaining high order of accuracy is to compute the integral term in the above equation. Especially for small values of x_i , there is little information useful for $u(a_l s + a_l - 1)$. To overcome this difficulty, we transfer the integral interval into an fixed interval $[-1, 1]$ by the following simple linear transformation

$$s_x(z) = \frac{x+1}{2}z + \frac{x-1}{2}, \quad z \in [-1, 1], \quad (2.3)$$

and then (2.2) becomes

$$u(x_i) - \sum_{l=1}^M \int_{-1}^1 K_l(x_i, z) u(a_l s_{x_i}(z) + a_l - 1) dz = g(x_i), \quad (2.4)$$

where $K_l(x, z) = \frac{x+1}{2} \hat{k}_l(x, s_x(z))$.

Next using a $N+1$ -point Gauss quadrature formula gives

$$u(x_i) - \sum_{l=1}^M \sum_{q=0}^N K_l(x_i, z_q) u(a_l s_{x_i}(z_q) + a_l - 1) w_q \approx g(x_i), \quad (2.5)$$

for $i = 0, 1, \dots, N$, where z_q are the $N+1$ Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto points, corresponding weight w_q , $q = 0, 1, \dots, N$. We use u_i to approximate the function value $u(x_i)$ and use

$$u^N(x) := \sum_{j=0}^N u_j F_j(x)$$

to approximate the function $u(x)$, where $F_j(x)$ is the j -th Lagrange basic function associated with $\{x_i\}_{i=0}^N$. Then the Chebyshev spectral method is to seek $u^N(x)$ such that $\{u_i\}_{i=0}^N$ satisfies the following equations for $i = 0, 1, \dots, N$,

$$u_i - \sum_{l=1}^M \sum_{q=0}^N K_l(x_i, z_q) u^N(a_l s_{x_i}(z_q) + a_l - 1) w_q = g(x_i), \quad (2.6)$$

which is equivalent to

$$u_i - \sum_{j=0}^N u_j \sum_{l=1}^M \sum_{q=0}^N \left(K_l(x_i, z_q) F_j(a_l s_{x_i}(z_q) + a_l - 1) \right) w_q = g(x_i). \quad (2.7)$$

We can also write the above equation in matrix form of $U - KU = G$, where

$$U := [u_0, u_1, \dots, u_N]^T, G := [g(x_0), g(x_1), \dots, g(x_N)]^T,$$

$$K := \left(\sum_{l=1}^M \sum_{k=0}^N \left(K_l(x_i, z_q) F_j(a_l s_{x_i}(z_q) + a_l - 1) \right) w_q \right)_{(N+1) \times (N+1)}.$$

3 Some spaces and lemmas

In this section we will introduce some spaces and lemmas that are prepared for the error analysis. First for non-negative integer m , we define

$$H_{\omega^{\alpha, \beta}}^m(-1, 1) := \{v : \partial_x^k v \in L_{\omega^{\alpha, \beta}}^2(-1, 1), 0 \leq k \leq m\},$$

with the norm

$$\|v\|_{H_{\omega^{\alpha, \beta}}^m(-1, 1)} := \left(\sum_{k=0}^m \|\partial_x^k v\|_{L_{\omega^{\alpha, \beta}}^2(-1, 1)}^2 \right)^{\frac{1}{2}}.$$

But in bounding the approximation error, only some of the L^2 -norms appearing on the right-hand side of the above norm enter into play. Thus, for a nonnegative integer N , it is convenient to introduce the semi-norm

$$|v|_{H_{\omega^{\alpha, \beta}}^{m; N}(-1, 1)} := \left(\sum_{k=\min(m, N+1)}^m \|\partial_x^k v\|_{L_{\omega^{\alpha, \beta}}^2(-1, 1)}^2 \right)^{\frac{1}{2}}.$$

Particularly when $\alpha = \beta = 0$, we denote $H_{\omega^{0, 0}}^{m; N}(-1, 1)$ by $H^{m; N}(-1, 1)$. When $\alpha = \beta = -\frac{1}{2}$, we denote $\omega^{-\frac{1}{2}, -\frac{1}{2}}$ by ω^c .

And the space $L^\infty(-1, 1)$ is the Banach space of the measurable functions $u : (-1, 1) \rightarrow \mathbb{R}$, which are bounded outside a set of measure zero, equipped the norm

$$\|u\|_{L^\infty(-1, 1)} := \text{ess sup}_{x \in (-1, 1)} |u(x)|.$$

Lemma 3.1 ([8, 15]). *Let $F_j(x)$, $j = 0, 1, \dots, N$ are the j -th Lagrange interpolation polynomials associated with $N+1$ Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points $\{x_j\}_{j=0}^N$. Then*

$$\|\mathcal{I}_N\|_{L^\infty(-1, 1)} := \max_{x \in [-1, 1]} \sum_{j=0}^N |F_j(x)| = O(\log N),$$

where \mathcal{I}_N is the interpolation operator associated with the $N+1$ Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points $\{x_i\}_{i=0}^N$, promptly

$$\mathcal{I}_N v := \sum_{i=0}^N v(x_i) F_i(x), \quad v \in C([-1, 1]).$$

Lemma 3.2 ([6, 17]). Assume that $u \in H_{\omega^c}^m(-1, 1)$, $m \geq 1$, then the following estimates hold

$$\begin{aligned} \|u - \mathcal{I}_N u\|_{L_{\omega^c}^2(-1, 1)} &\leq CN^{-m} |u|_{H_{\omega^c}^{m; N}(-1, 1)}, \\ \|u - \mathcal{I}_N u\|_{L^\infty(-1, 1)} &\leq CN^{\frac{1}{2}-m} |u|_{H_{\omega^c}^{m; N}(-1, 1)}. \end{aligned}$$

Lemma 3.3 ([6, 17]). Suppose $u \in H_{\omega^c}^m(-1, 1)$, $v \in H^m(-1, 1)$ for some $m \geq 1$ and $\psi \in P_N$, which denotes the space of all polynomials of degree not exceeding N . Then there exists a constant C independent of N such that

$$\begin{aligned} \left| \int_{-1}^1 u(x) \psi(x) \omega^c(x) dx - \sum_{j=0}^N u(x_j) \psi(x_j) \omega_j^c \right| &\leq CN^{-m} |u|_{H_{\omega^c}^{m; N}(-1, 1)} \|\psi\|_{L_{\omega^c}^2(-1, 1)}, \\ \left| \int_{-1}^1 v(x) \psi(x) dx - \sum_{j=0}^N v(z_j) \psi(z_j) \omega_j \right| &\leq CN^{-m} |v|_{H^{m; N}(-1, 1)} \|\psi\|_{L^2(-1, 1)}, \end{aligned}$$

where x_j is the $N+1$ Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto point, corresponding weight ω_j^c , $j=0, 1, \dots, N$ and z_j is $N+1$ Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto point, corresponding weight ω_j , $j=0, 1, \dots, N$.

Lemma 3.4 ([12, 18]). (Gronwall inequality) Assume that $u(x)$ is a nonnegative, locally integrable function defined on $[-1, 1]$, satisfying

$$u(x) \leq v(x) + B \int_{-1}^x u(\tau) d\tau,$$

where $B \geq 0$ is a constant and $v(x)$ is integrable function. Then there exists a constant C such that

$$u(x) \leq v(x) + C \int_{-1}^x v(\tau) d\tau,$$

and

$$\|u(x)\|_{L^\infty(-1, 1)} \leq C \|v(x)\|_{L^\infty(-1, 1)}.$$

Lemma 3.5. Suppose $0 \leq B_1, B_2, \dots, B_M < +\infty$. If a nonnegative integrable function $e(x)$ satisfies

$$e(x) \leq v(x) + \sum_{l=1}^M B_l \int_{-1}^x e(a_l s + a_l - 1) ds,$$

where $v(x)$ is a nonnegative function too. Then there exists a constant C such that

$$e(x) \leq v(x) + C \int_{-1}^x v(t) dt, \quad \|e(x)\|_{L^\infty(-1, 1)} \leq C \|v(x)\|_{L^\infty(-1, 1)}.$$

Lemma 3.6 ([13]). For all measurable function $f \geq 0$, the following generalized Hardy's inequality

$$\left(\int_a^b |(Tf)(x)|^q \omega_1(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b |f(x)|^p \omega_2(x) dx \right)^{\frac{1}{p}}$$

holds if and only if

$$\sup_{a < x < b} \left(\int_x^b \omega_1(t) dt \right)^{\frac{1}{q}} \left(\int_a^x \omega_2(t)^{1-p'} dt \right)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1},$$

for the case $1 < p \leq q < \infty$. Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x,t) f(t) dt$$

with $k(x,t)$ a given kernel, ω_1, ω_2 weight functions, and $-\infty \leq a < b \leq +\infty$.

Lemma 3.7 ([13, 16]). For all bounded function $v(x)$, there exists a constant C independent of v such that

$$\sup_N \|\mathcal{I}_N v\|_{L^2_{\omega^c}(-1,1)} \leq C \|v\|_{L^\infty(-1,1)}.$$

4 Error analysis

Now we turn to give the main result of the article. Our goal is to show the rate of convergence decay exponentially in the infinity space and the Chebyshev weighted Hilbert space. Firstly we carry out our analysis in L^∞ space.

Theorem 4.1. Assume that $u(x)$ is the exact solution of (2.1) and $u^N(x)$ is the approximate solution achieved by Chebyshev spectral method from (2.6). Then for N sufficiently large, we get

$$\|u(x) - u^N(x)\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} (|u|_{H^{m;N}_{\omega^c}(-1,1)} + K^* \|u\|_{L^\infty(-1,1)}), \tag{4.1}$$

where

$$K^* := \sum_{l=1}^M \max_{-1 \leq x \leq 1} |K_l(x, \cdot)|_{H^{m;N}(-1,1)}.$$

Proof. Make subtraction from (2.4) to (2.6) and we have

$$\begin{aligned} & u(x_i) - u_i \tag{4.2} \\ &= \sum_{l=1}^M \int_{-1}^1 K_l(x_i, z) u(a_l s_{x_i}(z) + a_l - 1) dz - \sum_{l=1}^M \sum_{q=0}^N K_l(x_i, z_q) u^N(a_l s_{x_i}(z_q) + a_l - 1) \omega_q \\ &= \sum_{l=1}^M \left(\int_{-1}^1 K_l(x_i, z) u(a_l s_{x_i}(z) + a_l - 1) dz - \int_{-1}^1 K_l(x_i, z) u^N(a_l s_{x_i}(z) + a_l - 1) dz \right) \\ &\quad + \sum_{l=1}^M \left(\int_{-1}^1 K_l(x_i, z) u^N(a_l s_{x_i}(z) + a_l - 1) dz - \sum_{q=0}^N K_l(x_i, z_q) u^N(a_l s_{x_i}(z_q) + a_l - 1) \omega_q \right). \end{aligned}$$

If we let $e(x) = u(x) - u^N(x)$ and (4.2) turns into

$$u(x_i) - u_i = \sum_{l=1}^M \int_{-1}^1 K_l(x_i, z) e(a_l s_{x_i}(z) + a_l - 1) dz + I_1(x_i), \quad (4.3)$$

where

$$I_1(x) = \sum_{l=1}^M \left(\int_{-1}^1 K_l(x_i, z) u^N(a_l s_{x_i}(z) + a_l - 1) dz - \sum_{q=0}^N K_l(x_i, z_q) u^N(a_l s_{x_i}(z_q) + a_l - 1) \omega_q \right).$$

To estimate $I_1(x)$, using Lemma 3.3, we deduce that

$$|I_1(x)| \leq \sum_{l=1}^M C_M N^{-m} \|K_l(x, \cdot)\|_{H^{m;N}(-1,1)} \|u^N(a_l(s_x(\cdot)) + a_l - 1)\|_{L^2(-1,1)}.$$

We multiply $F_i(x)$ on both sides of (4.3), sum up from $i=0$ to N and get

$$\mathcal{I}_N u - u^N(x) = \sum_{l=1}^M \mathcal{I}_N \int_{-1}^1 K_l(x, z) e(a_l s_x(z) + a_l - 1) dz + \mathcal{I}_N I_1(x) = e(x) + \mathcal{I}_N u - u,$$

subsequently,

$$e(x) = \sum_{l=1}^M \int_{-1}^1 K_l(x, z) e(a_l s_x(z) + a_l - 1) dz + \sum_{j=0}^2 J_j(x),$$

where

$$J_0(x) = u - \mathcal{I}_N u, J_1(x) = \sum_{i=0}^N I_1(x_i) F_i(x),$$

$$J_2(x) = \sum_{l=1}^M \left(\mathcal{I}_N \int_{-1}^1 K_l(x, z) e(a_l s_x(z) + a_l - 1) dz - \int_{-1}^1 K_l(x, z) e(a_l s_x(z) + a_l - 1) dz \right).$$

We rewrite $e(x)$ as follows by using the inverse process of (2.3)

$$e(x) = \sum_{l=1}^M \int_{-1}^x \hat{k}_l(x, s) e(a_l s + a_l - 1) ds + \sum_{j=0}^2 J_j(x),$$

then we have

$$|e(x)| \leq \sum_{l=1}^M B_l \int_{-1}^x |e(a_l s + a_l - 1)| ds + v(x),$$

where

$$B_l = \max_{(x,s) \in \Omega_1} |\hat{k}_l(x, s)|, v(x) = \left| \sum_{j=0}^2 J_j(x) \right|.$$

Using Lemma 3.5, we obtain

$$\|e(x)\| \leq v(x) + C \int_{-1}^x v(t) dt, \tag{4.4}$$

$$\|e\|_{L^\infty(-1,1)} \leq C \sum_{j=0}^2 \|J_j\|_{L^\infty(-1,1)}. \tag{4.5}$$

Now we come to estimate each $J_j(x)$. First to reckon $\|J_0\|_{L^\infty(-1,1)}$, with the help of Lemma 3.2, we get

$$\|J_0\|_{L^\infty(-1,1)} = \|u - \mathcal{I}_N u\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} |u|_{H_{\omega^c}^{m;N}(-1,1)}.$$

Then for the evaluate of $J_1(x)$, using Lemma 3.3, we know that

$$\max_{-1 \leq x \leq 1} |I_1(x)| \leq CN^{-m} K^* \|u^N\|_{L^\infty(-1,1)} \leq CN^{-m} K^* (\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}),$$

together with Lemma 3.1, we have

$$\begin{aligned} \|J_1(x)\|_{L^\infty(-1,1)} &\leq \|\mathcal{I}_N\|_{L^\infty(-1,1)} \|I_1(x)\|_{L^\infty(-1,1)} \\ &\leq CN^{-m} (\log N) K^* (\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}). \end{aligned}$$

In order to bound $J_2(x)$, apply the second conclusion in Lemma 3.2 and let $m = 1$ and we yield

$$\begin{aligned} &\|J_2(x)\|_{L^\infty(-1,1)} \\ &\leq \sum_{l=1}^M C_l N^{-\frac{1}{2}} \left| \int_{-1}^1 K_l(x,z) e(a_l s_x(z) + a_l - 1) dz \right|_{H_{\omega^c}^{1;N}(-1,1)} \\ &= \sum_{l=1}^M C_l N^{-\frac{1}{2}} \left| \int_{-1}^x K_l(x,s) e(a_l s + a_l - 1) ds \right|_{H_{\omega^c}^{1;N}(-1,1)} \\ &= \sum_{l=1}^M C_l N^{-\frac{1}{2}} \left\| K_l(x,x) e(a_l x + a_l - 1) + \int_{-1}^x e(a_l s + a_l - 1) \frac{\partial}{\partial x} K_l(x,s) ds \right\|_{L_{\omega^c}^2(-1,1)} \\ &\leq CN^{-\frac{1}{2}} \|e(x)\|_{L_{\omega^c}^2(-1,1)} \leq CN^{-\frac{1}{2}} \|e(x)\|_{L^\infty(-1,1)}. \end{aligned}$$

From what has been discussed above, we yield

$$\|e\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} |u|_{H_{\omega^c}^{m;N}(-1,1)} + CN^{-m} (\log N) K^* \|u\|_{L^\infty(-1,1)}. \tag{4.6}$$

Since for N sufficiently large, $\log N < N^{\frac{1}{2}}$, therefore we get the desired estimate

$$\|e\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} (|u|_{H_{\omega^c}^{m;N}(-1,1)} + K^* \|u\|_{L^\infty(-1,1)}).$$

So we finish the proof of Theorem 4.1. □

Next we will give the error analysis in $L^2_{\omega^c}$ space.

Theorem 4.2. *Assume that $u(x)$ is the exact solution of (2.1) and $u^N(x)$ is the approximate solution obtained by Chebyshev spectral method from (2.6). Then for N sufficiently large, we get*

$$\|u - u^N\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m}(K^* + 1) \left(|u|_{H^{m;N}_{\omega^c}(-1,1)} + \|u'\|_{L^2_{\omega^c}(-1,1)} + \|u\|_{L^\infty(-1,1)} \right). \tag{4.7}$$

Proof. The same method followed as the first part of Theorem 4.1 to (4.4), and by Lemma 3.6, we have

$$|e(x)|_{L^2_{\omega^c}(-1,1)} \leq C \sum_{j=0}^2 \|J_j\|_{L^2_{\omega^c}(-1,1)}. \tag{4.8}$$

Now we come to bound each term of the right-hand side of (4.8). First applying Lemma 3.2 to $J_0(x)$ gives:

$$\|J_0(x)\|_{L^2_{\omega^c}(-1,1)} = \|u - \mathcal{I}_N u\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m} |u|_{H^{m;N}_{\omega^c}(-1,1)}. \tag{4.9}$$

To estimate $\|J_1(x)\|_{L^2_{\omega^c}(-1,1)}$, we get the following result by Lemma 3.7,

$$\|J_1(x)\|_{L^2_{\omega^c}(-1,1)} \leq C \max_{-1 \leq x \leq 1} |I_1(x)| \leq CN^{-m} K^* (\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}).$$

Furthermore, if we let $m = 1$ in Theorem 4.1, we have

$$\|e\|_{L^\infty(-1,1)} \leq C (\|u'\|_{L^2_{\omega^c}(-1,1)} + K^* \|u\|_{L^\infty(-1,1)}).$$

Consequently,

$$\|J_1(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m} K^* \left(\|u'\|_{L^2_{\omega^c}(-1,1)} + (K^* + 1) \|u\|_{L^\infty(-1,1)} \right). \tag{4.10}$$

To bound $\|J_2(x)\|_{L^2_{\omega^c}(-1,1)}$, with the help from the first conclusion in Lemma 3.2, if we let $m = 1$, as the same analysis in Theorem 4.1 for $\|J_2(x)\|_{L^\infty(-1,1)}$, we can deduce that

$$\|J_2(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-1} \|e(x)\|_{L^\infty(-1,1)},$$

and furthermore using the convergence result in Theorem 4.1, we obtain that

$$\|J_2(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{\frac{1}{2}-m} (|u|_{H^{m;N}_{\omega^c}(-1,1)} + K^* \|u\|_{L^\infty(-1,1)}). \tag{4.11}$$

Combining (4.8)–(4.11), we get the desired conclusion

$$\|u - u^N\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m}(K^* + 1) \left(|u|_{H^{m;N}_{\omega^c}(-1,1)} + \|u'\|_{L^2_{\omega^c}(-1,1)} + \|u\|_{L^\infty(-1,1)} \right). \quad \square$$

5 A numerical example

In this section, we will give numerical examples to demonstrate the theoretical result proposed in Section 4. First of all, we consider (2.1) with

$$u(x) = e^{4x}, \quad M = 1, \quad \hat{k}_1(x, s) = e^{-xs}, \quad a_1 = 1, \quad g(x) = e^{4x} + \frac{1}{x+4}(e^{x(x+4)} - e^{-(x+4)}).$$

The corresponding errors versus several values of N are displayed in Table 1. It is easy to find that the errors both decay in L^∞ and $L^2_{\omega^c}$ norms.

Table 1: The errors versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms.

N	6	8	10	12	14
$L^\infty - error$	3.35e-002	1.25e-003	3.31e-005	6.53e-007	1.01e-008
$L^2_{\omega^c} - error$	4.04e-002	1.46e-003	3.92e-005	7.92e-007	1.24e-008
N	16	18	20	22	24
$L^\infty - error$	1.25e-010	1.26e-012	4.97e-014	3.55e-014	4.26e-014
$L^2_{\omega^c} - error$	1.55e-010	1.57e-012	8.00e-014	8.03e-014	8.18e-014

Moreover we give two graphs below. The left graph plots the errors for $6 \leq N \leq 24$ in both L^∞ and $L^2_{\omega^c}$ norms. The approximate solution ($N = 24$) and the exact solution are displayed in the right graph.

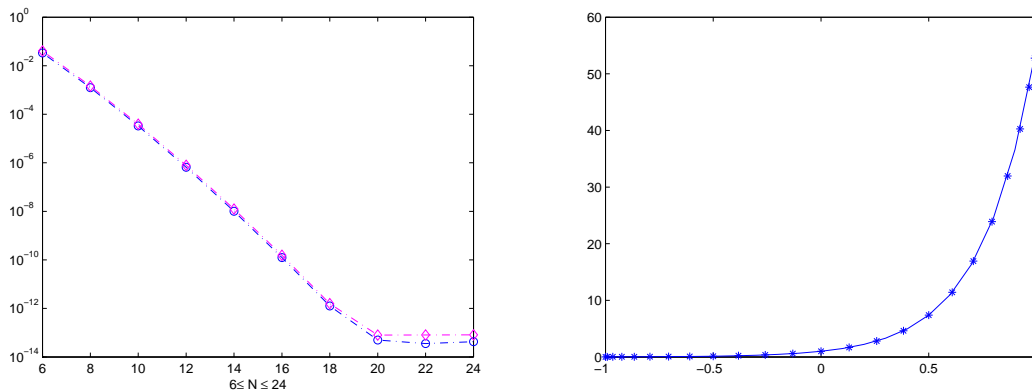


Figure 1: The errors versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms (left). Comparison between approximate solution and the exact solution (right).

This example has appeared in [18]. Comparing the errors in [18] and ours, one is easy to find that the accuracy obtained by Chebyshev is higher than the Legendre spectral method. Without loss of generality, we will give another example to confirm our theoretical result.

Table 2: The errors versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms.

N	2	4	6	8	10
L^∞ -error	7.44e-002	2.52e-003	1.00e-005	1.41e-008	1.99e-011
$L^2_{\omega^c}$ -error	8.36e-002	2.31e-003	8.02e-006	1.20e-008	1.88e-011
N	12	14	16	18	20
L^∞ -error	3.90e-014	4.77e-015	2.16e-015	3.86e-015	4.80e-015
$L^2_{\omega^c}$ -error	3.21e-014	4.91e-015	2.63e-015	3.80e-015	4.19e-015

Now we consider (1.1) with $T=2$, $M=2$, $a_1=\frac{1}{3}$, $a_2=\frac{1}{2}$, $k_1(x,\xi)=-(t-\xi)$, $k_2(x,\xi)=-(t+\xi)$, and

$$f(t) = \cos t + 4t \sin \frac{t}{2} + 4 \cos \frac{t}{2} - 9 \cos \frac{t}{3} + 5.$$

The exact solution is $y(t) = \cos t$.

There are also two graphs below. In the same way, the errors for $2 \leq N \leq 20$ in both L^∞ and $L^2_{\omega^c}$ norms are displayed in the left graph. The numerical ($N=20$) and the exact solution are displayed in the right graph. Moreover, the corresponding errors with several values of N are displayed in Table 2. As expected, the errors decay exponentially which are found in excellent agreement.

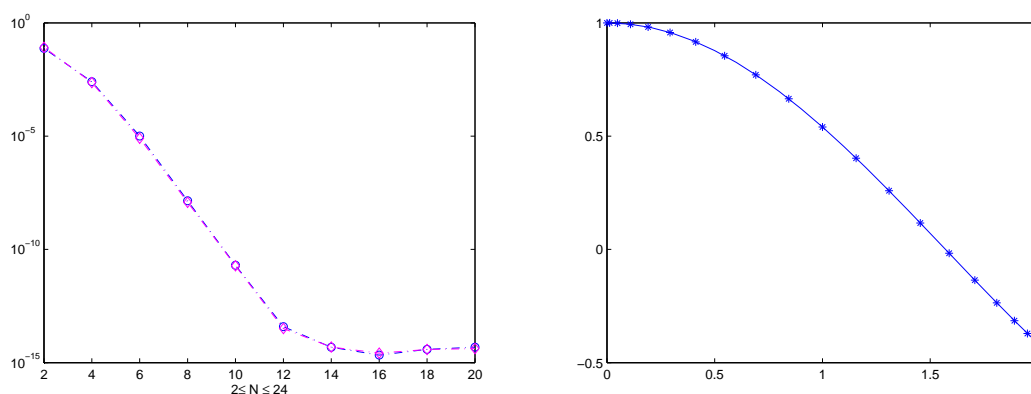


Figure 2: The errors versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms (left). Comparison between approximate solution and the exact solution (right).

6 Conclusion

In this paper, we successfully provide a rigorous error analysis for the Volterra integral equation with multiple delays by Chebyshev spectral method. We get the conclusion that the numerical error both decay exponentially in L^∞ and $L^2_{\omega^c}$ norms. Moreover the accuracy obtained by our paper is higher than the Legendre spectral method.

Acknowledgments

This work was supported by NSF of China No. 11671157, No.91430104 and No. 11626074 and Hanshan Normal University project No. 201404 and No. Z16027.

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