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Hexagonal Fourier-Galerkin Methods for the Two-Dimensional Homogeneous Isotropic Decaying Turbulence

Huiyuan Li*

State Key Laboratory of Computer Science, Institute of Software Chinese Academy of Sciences, Beijing 100190, China.

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Abstract. In this paper, we propose two hexagonal Fourier-Galerkin methods for the direct numerical simulation of the two-dimensional homogeneous isotropic decaying turbulence. We first establish the lattice Fourier analysis as a mathematical foundation. Then a universal approximation scheme is devised for our hexagonal Fourier-Galerkin methods for Navier-Stokes equations. Numerical experiments mainly concentrate on the decaying properties and the self-similar spectra of the two-dimensional homogeneous turbulence at various initial Reynolds numbers with an initial flow field governed by a Gaussian-distributed energy spectrum. Numerical results demonstrate that both the hexagonal Fourier-Galerkin methods are as efficient as the classic square Fourier-Galerkin method, while provide more effective statistical physical quantities in general.

AMS subject classifications: 65M70, 65T50, 76F05, 76F65, 76M22

Key words: Fourier-Galerkin methods, hexagonal lattices, homogeneous isotropic turbulence, direct numerical simulation.

1 Introduction

The study of two-dimensional homogeneous isotropic decaying turbulence presents several interests, because of not only its applications to geophysics and astrophysics, but also the basic understanding to hydrodynamic turbulence. Far from being a simplified version of the three-dimensional problem, two-dimensional turbulence presents a rich panorama of new phenomena [3]. There are many remarkable characteristics in 2D turbulence fields, such as coherent structures [1, 13, 19], inverse energy cascade and direct enstrophy cascade [2,14,15]. The inverse energy cascade indicates that the energy is transferred in the inviscid limit from small scales to large scales instead of from large scales

*Corresponding author. *Email address:* `huiyuan@iscas.ac.cn` (H. Li)

to small scales as in the three dimensions. While the enstrophy exhibits a direct cascade process as energy cascade in 3D. The double cascade makes the study of 2D turbulence even more complicated and challenging than that in 3D.

The Navier-Stokes equation for the two-dimensional turbulence may be written in the velocity-vorticity form

$$\frac{\partial \omega}{\partial t}+\nabla\cdot(\boldsymbol{u}\omega)=\nu\Delta\omega, \tag{1.1}$$

$$\omega=\nabla\times\boldsymbol{u},\quad \nabla\cdot\boldsymbol{u}=0, \tag{1.2}$$

where the kinematic viscosity ν can be interpreted as the reciprocal of the Reynolds number. Let ψ be the stream function. Then

$$\boldsymbol{u}=(\partial_{x_2}\psi,-\partial_{x_1}\psi),\qquad \omega=-\Delta\psi\quad\text{and}\quad \nabla\cdot(\boldsymbol{u}\omega)=\frac{\partial\psi}{\partial x_2}\frac{\partial\omega}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\omega}{\partial x_2}.$$

Thus, the Navier-Stokes equation for the two-dimensional turbulence can also be written as the stream function-vorticity equation,

$$\begin{aligned}&\frac{\partial\omega}{\partial t}+\frac{\partial\psi}{\partial x_2}\frac{\partial\omega}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\omega}{\partial x_2}=\nu\Delta\omega,\\&-\Delta\psi=\omega.\end{aligned}$$

Owing to (1.2), the Jacobian can be reformulated as $\nabla\cdot(\boldsymbol{u}\omega)=\partial_{x_2}\partial_{x_1}(u_2^2-u_1^2)-(\partial_{x_2}^2-\partial_{x_1}^2)(u_1u_2)$, which allows an efficient algorithm for the fast evaluation.

Navier-Stokes equation in two dimensions has the appealing feature to be less demanding on a computational level than the three-dimensional case, allowing to reach relatively high *Re* numbers in direct numerical simulation (DNS). As a powerful tool, DNS provides some useful theory check and inspires the deeper thoughts of the statistical theory. The spectral method has been becoming very popular in the research of highly accurate numerical simulations since the pioneer work of Orszag and Patterson [21]. DNS of 2D turbulence followed an explosive trend in the early stage with an increasing resolution from 256^2 to 4096^2 or even higher [2,4,5,11,18,23]. Up to the present, DNS of 2D homogeneous isotropic decaying turbulence are usually been carried out with a millennial resolution using variants of Fourier spectral or pseudospectral methods with the tensorial Fourier basis functions $\{e^{i(k_1x_1+k_2x_2)}\}_{-n/2\le k_1,k_2\le n/2-1}$ subject to periodic boundary conditions [5,18,23].

From a general view, the classic Fourier basis functions are just samples of the complex exponential $e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}$ on the rectangular lattice $\mathbb{Z}^2$ in the frequency space, i.e, with the wave vectors $\boldsymbol{\xi}=\boldsymbol{k}\in\mathbb{Z}^2$. Nevertheless, they actually form a complete orthogonal system on the Voronoi cell $\{\boldsymbol{x}\in\mathbb{R}^2:-\pi\le x_1,x_2\le\pi\}$ of the dual lattice $2\pi\mathbb{Z}^2$ in the physical space, which can represent the solution of (1.1)-(1.2) by a Fourier series with its coefficients to be determined. Inspired by the success of the plane-wave method in quantum physics, one can also choose a proper lattice L^* (e.g. $2\pi\boldsymbol{A}^{-1}\mathbb{Z}^2$ with certain nonsingular matrix $\boldsymbol{A}$)