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Spectral Method for the Black-Scholes Model of American Options Valuation

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Abstract. In this paper, we devote ourselves to the research of numerical methods for American option pricing problems under the Black-Scholes model. The optimal exercise boundary which satisfies a nonlinear Volterra integral equation is resolved by a high-order collocation method based on graded meshes. For the other spatial domain boundary, an artificial boundary condition is applied to the pricing problem for the effective truncation of the semi-infinite domain. Then, the front-fixing and stretching transformations are employed to change the truncated problem in an irregular domain into a one-dimensional parabolic problem in $[-1,1]$. The Chebyshev spectral method coupled with fourth-order Runge-Kutta method is proposed for the resulting parabolic problem related to the options. The stability of the semi-discrete numerical method is established for the parabolic problem transformed from the original model. Numerical experiments are conducted to verify the performance of the proposed methods and compare them with some existing methods.

AMS subject classifications: 35A35, 90A09, 65K10, 65M12, 65M60

Key words: American option pricing, Black-Scholes model, optimal exercise boundary, front-fixing, Chebyshev spectral method, Runge-Kutta method

1 Introduction

Options as a kind of important financial derivatives have a wide range of applications. The option pricing problem, especially the American option pricing problem, attracts the attention of more and more financial practitioners. The distinctive feature of an American option is its early exercise privilege, that is, the holder is endowed with the additional right to exercise the option at any time prior to the date of expiration, which makes an American option worth more than its European counterpart. Because of its early exercise

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privilege, American option pricing problem exists an optimal exercise boundary [3,26], which makes American pricing problem a nonlinear problem, and no analytical solution exists comparing to that of European options.

The researches of American option pricing problems have been extensively developed in recent decades. Traditionally, there are two ways—the analytical method and the numerical method. For analytical results, we refer to [1,9,10,13,25,33]. They managed to present the solutions in a closed form depending on the optimal exercise boundary, then the option value was determined as long as the optimal exercise boundary was given. However it was not known actually in practice. For numerical aspects, it can be divided into two categories in general, one is based on Monte Carlo approach [11,28,30], the other is based on partial differential equation (PDE) approach [2,3,19,20,22]. We would like to further adopt the PDE approach in this paper, since Monte Carlo method requires demanding computational resource due to its slow convergence.

The Black-Scholes equation is one of the most effective PDE models [5,21]. And numerical methods are of popular use and frequently resorted for the Black-Scholes model among financial practitioners. For example, lattice tree methods, finite difference methods and finite element methods have been developed and extensively studied in recent decades. Cox, Ross and Rubinstein firstly introduced the binomial model to price American options in their seminal paper [15]. Later, Amin and Khanna showed the convergence of the binomial method in [4]. Finite difference methods have been developed and discussed for a long time for American option pricing problem [8,19]. One may refer to [22] for the convergence analysis. Recently, finite element methods [3,20] have attracted more interest in this field for its solid theoretical framework, in particularly its efficiency and variability. Interested readers may refer to [26] and the references therein for a complete survey. In this paper, we discuss the Chebyshev spectral method to the same problem, which turns to be efficient and comparable to finite element method.

There exist four main challenges for the numerical treatment of the American option pricing problem:

- The optimal exercise boundary is unknown, which satisfies a highly nonlinear equation, so it is not easy to get a fine resolution of optimal exercise boundary.
- For the other boundary, since we cannot adopt numerical methods directly to unbounded domain, how to truncate the unbounded domain and control the truncated errors are the key issues.
- The solving domain is an irregular domain, and the partition has to be reconstructed for each time order in order to resolve the option values exactly.
- The last one is how to choose an efficient numerical method to solve the problem fast and accurately.

About the first challenge, Cox [15,23] has proved that the optimal exercise boundary satisfied a nonlinear Volterra integral equation, and Ma et al. [29] have solved the same