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Uniformly Accurate Multiscale Time Integrators for Highly Oscillatory Second Order Differential Equations

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Abstract. In this paper, two multiscale time integrators (MTIs), motivated from two types of multiscale decomposition by either frequency or frequency and amplitude, are proposed and analyzed for solving highly oscillatory second order differential equations with a dimensionless parameter $0<\varepsilon\le 1$. In fact, the solution to this equation propagates waves with wavelength at $O(\varepsilon^2)$ when $0<\varepsilon\ll 1$, which brings significantly numerical burdens in practical computation. We rigorously establish two independent error bounds for the two MTIs at $O(\tau^2/\varepsilon^2)$ and $O(\varepsilon^2)$ for $\varepsilon\in(0,1]$ with $\tau>0$ as step size, which imply that the two MTIs converge uniformly with linear convergence rate at $O(\tau)$ for $\varepsilon\in(0,1]$ and optimally with quadratic convergence rate at $O(\tau^2)$ in the regimes when either $\varepsilon=O(1)$ or $0<\varepsilon\le\tau$. Thus the meshing strategy requirement (or ε-scalability) of the two MTIs is $\tau=O(1)$ for $0<\varepsilon\ll 1$, which is significantly improved from $\tau=O(\varepsilon^3)$ and $\tau=O(\varepsilon^2)$ requested by finite difference methods and exponential wave integrators to the equation, respectively. Extensive numerical tests and comparisons with those classical numerical integrators are reported, which gear towards better understanding on the convergence and resolution properties of the two MTIs. In addition, numerical results support the two error bounds very well.

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1 Introduction

This paper is devoted to the study of numerical solutions of the following highly oscillatory second order differential equations (ODEs)

$$\begin{cases} \varepsilon^2 \ddot{\mathbf{y}}(t) + A\mathbf{y}(t) + \frac{1}{\varepsilon^2}\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t)) = 0, \quad t>0, \\ \mathbf{y}(0) = \Phi_1, \quad \dot{\mathbf{y}}(0) = \frac{\Phi_2}{\varepsilon^2}. \end{cases} \tag{1.1}$$

Here t is time, $\mathbf{y} := \mathbf{y}(t) = (y_1(t),\cdots,y_d(t))^T \in \mathbb{C}^d$ is a complex-valued vector function with d a positive integer, $\dot{\mathbf{y}}$ and $\ddot{\mathbf{y}}$ refer to the first and second order derivatives of $\mathbf{y}$, respectively, $0<\varepsilon\leq 1$ is a dimensionless parameter which can be very small in some limit regimes, $A\in\mathbb{R}^{d\times d}$ is a symmetric nonnegative definite matrix, Φ_1, $\Phi_2\in\mathbb{C}^d$ are two given initial data at $O(1)$ in terms of $0<\varepsilon\ll 1$, and $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}),\cdots,f_d(\mathbf{y}))^T:\ \mathbb{C}^d\to\mathbb{C}^d$ describes the nonlinear interaction and it is independent of ε. The *gauge invariance* implies that $\mathbf{f}(\mathbf{y})$ satisfies the following relation [34]

$$\mathbf{f}(e^{is}\mathbf{y}) = e^{is}\mathbf{f}(\mathbf{y}), \quad \forall s\in\mathbb{R}. \tag{1.2}$$

We remak that when the initial data $\Phi_1,\Phi_2\in\mathbb{R}^d$ and $\mathbf{f}(\mathbf{y}):\ \mathbb{R}^d\to\mathbb{R}^d$, then the solution $\mathbf{y}\in\mathbb{R}^d$ is real-valued. In this case, the gauge invariance condition (1.2) for the nonlinearity in (1.1) is no longer needed.

The above problem is motivated from our recent numerical study of the nonlinear Klein-Gordon equation in the nonrelativistic limit regime [5,33,34], where $0<\varepsilon\ll 1$ is scaled to be inversely proportional to the speed of light. In fact, it can be viewed as a model resulting from a semi-discretization in space, e.g., by finite difference or spectral discretization with a fixed mesh size (see detailed equations (3.3) and (3.19) in [5]), to the nonlinear Klein-Gordon equation. In order to propose new multiscale time integrators (MTIs) and compare with those classical numerical integrators including finite difference methods [5,16,32,39] and exponential wave integrators [19,25–27,36] efficiently, we thus focus on the above second order differential equations instead of the original nonlinear Klein-Gordon equation. The solution to (1.1) propagates high oscillatory waves with wavelength at $O(\varepsilon^2)$ and amplitude at $O(1)$. To illustrate this, Fig. 1 shows the solutions of (1.1) with $d=2$, $f_1(y_1,y_2) = y_1^2 y_2$, $f_2(y_1,y_2) = y_2^2 y_1$, $A = \mathrm{diag}(2,2)$, $\Phi_1 = (1,0.5)^T$ and $\Phi_2 = (1,2)^T$ for different ε. The highly oscillatory nature of solutions to (1.1) causes severe burdens in practical computation, making the numerical approximation extremely challenging and costly in the regime of $0<\varepsilon\ll 1$.

For the global well-posedness of the model problem (1.1), we refer to [29,30] and references therein. For simplicity of notations, we will present our methods and comparison