J. Math. Study
doi: 10.4208/jms.v47n2.14.02

Vol. **47**, No. 2, pp. 151-172
June 2014

A Study on the Conditioning of Finite Element Equations with Arbitrary Anisotropic Meshes via a Density Function Approach

Lennard Kamenski[1,*] and Weizhang Huang[2]

[1] *Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany.*
[2] *Department of Mathematics, the University of Kansas, Lawrence, Kansas, USA.*

Received 18 September 2013; Accepted 5 December 2013

Abstract. The linear finite element approximation of a general linear diffusion problem with arbitrary anisotropic meshes is considered. The conditioning of the resultant stiffness matrix and the Jacobi preconditioned stiffness matrix is investigated using a density function approach proposed by Fried in 1973. It is shown that the approach can be made mathematically rigorous for general domains and used to develop bounds on the smallest eigenvalue and the condition number that are sharper than existing estimates in one and two dimensions and comparable in three and higher dimensions. The new results reveal that the mesh concentration near the boundary has less influence on the condition number than the mesh concentration in the interior of the domain. This is especially true for the Jacobi preconditioned system where the former has little or almost no influence on the condition number. Numerical examples are presented.

AMS subject classifications: 65N30, 65N50, 65F35, 65F15

Chinese Library Classifications: O241.82, O242.21

Key words: Conditioning, finite element, anisotropic diffusion, anisotropic mesh, stiffness matrix, extreme eigenvalue, Jacobi preconditioning, diagonal scaling.

---

1 Introduction

Mesh adaptation is a common tool for use in the numerical solution of partial differential equations (PDEs) to enhance computational efficiency. It often results in nonuniform meshes whose elements vary significantly in size and shape from place to place on the

*Corresponding author. *Email addresses:* `kamenski@wias-berlin.de` (L. Kamenski), `whuang@ku.edu` (W. Huang)

physical domain. Nonuniform meshes could lead to ill-conditioned linear systems and their solution may deteriorate the efficiency of the entire computation. It is thus important in practice as well as in theory to understand how mesh nonuniformity affects the conditioning of linear systems resulting from discretization of PDEs on nonuniform meshes.

The issue has been studied by a number of researchers mostly for the linear finite element approximation of the Laplace operator or a general diffusion operator by developing bounds on the extremal eigenvalues on the resultant stiffness matrix, e.g., see [4, 6, 7, 13–15] for second-order elliptic PDEs or [2, 3, 9] for a more general setting of elliptic bilinear forms on Sobolev spaces of real index $m \in [-1,1]$.

The estimation of the largest eigenvalue is well understood and it is easy to show that the largest eigenvalue is bounded by a multiple (with a constant depending on mesh connectivity) of the maximum of the largest eigenvalues of the local stiffness matrices [7]. Moreover, the largest diagonal entry of the stiffness matrix is a good estimate for the largest eigenvalue: it is tight within a factor of $d+1$ for any mesh, where d is the dimension of the physical domain [13]. Sharp estimates in terms of mesh geometry are available for both isotropic [3, 6, 9, 14] and anisotropic [13] diffusion.

The estimation of the smallest eigenvalue is more challenging. Currently there are two approaches for this purpose. The first approach utilizes Sobolev's inequality and was first used by Bank and Scott [4] for the Laplace operator with isotropic meshes in $d \geq 2$ dimensions. They developed a lower bound on the smallest eigenvalue of a diagonally scaled stiffness matrix and showed that the condition number of the scaled stiffness matrix is comparable to that with a uniform mesh. A similar result for elliptic bilinear forms on Sobolev spaces of real index $m \in [-1,1]$ with shape-regular meshes in $d \geq 2$ dimensions was derived by Ainsworth, McLean, and Tran [2, 3]. Their result was later generalized to locally quasi-uniform meshes[†] in $d \geq 3$ dimensions by Graham and McLean [9]. Recently, Kamenski, Huang, and Xu [13] derived a similar bound for second-order elliptic PDEs which is valid for arbitrary meshes (i.e., without imposing any conditions on the mesh regularity) for any d; the established bound for the condition number depends on three factors, one representing the condition number of the linear finite element equations for the Laplace operator on a uniform mesh and the other factors arising from the nonuniformity of the mesh viewed in the metric specified by the inverse of the diffusion matrix $\mathbb{D}$ ($\mathbb{D}^{-1}$-*nonuniformity*) and the mesh nonuniformity in volume measured in the Euclidean metric (*volume-nonuniformity*). Further, it was shown in [13] that the Jacobi preconditioning — an optimal diagonal scaling for a symmetric positive definite sparse matrix — eliminates the effect of the mesh volume-nonuniformity and reduces the effect of the mesh $\mathbb{D}^{-1}$-nonuniformity. This result can be seen as a further generalization of [3, 4, 9] towards arbitrary anisotropic meshes and general diffusion coefficients.

In the second approach (hereafter referred to as *the density function approach*), a lower bound on the smallest eigenvalue of the stiffness matrix is obtained through a lower

---

[†]They are meshes where neighboring elements always comparable size and shape.