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Regularity of Positive Solutions for an Integral System on Heisenberg Group

Weiyang Chen and Xiaoli Chen*

Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China.

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Abstract. In this paper, we are concerned with the properties of positive solutions of the following nonlinear integral systems on the Heisenberg group $\mathbb{H}^n$,

$$\begin{cases} u(x)=\int_{\mathbb{H}^n}\frac{v^q(y)w^r(y)}{|x^{-1}y|^\alpha|y|^\beta}dy,\\ v(x)=\int_{\mathbb{H}^n}\frac{u^p(y)w^r(y)}{|x^{-1}y|^\alpha|y|^\beta}dy,\\ w(x)=\int_{\mathbb{H}^n}\frac{u^p(y)v^q(y)}{|x^{-1}y|^\alpha|y|^\beta}dy, \end{cases} \tag{0.1}$$

for $x\in\mathbb{H}^n$, where $0<\alpha<Q=2n+2$, $n\geq 3$, $\beta\geq 0$, $\alpha+\beta<Q$, and $p,q,r>1$ satisfying $\frac{1}{p+1}+\frac{1}{q+1}+\frac{1}{r+1}=\frac{Q+\alpha+\beta}{Q}$. We show that positive solution triples $(u,v,w)\in L^{p+1}(\mathbb{H}^n)\times L^{q+1}(\mathbb{H}^n)\times L^{r+1}(\mathbb{H}^n)$ are bounded and they converge to zero when $|x|\to\infty$.

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1 Introduction

In this paper, we investigate the properties of positive solutions of the following nonlinear integral system on the Heisenberg group

$$\begin{cases} u(x)=\int_{\mathbb{H}^n}\frac{v^q(y)w^r(y)}{|x^{-1}y|^\alpha|y|^\beta}dy,\\ v(x)=\int_{\mathbb{H}^n}\frac{u^p(y)w^r(y)}{|x^{-1}y|^\alpha|y|^\beta}dy,\\ w(x)=\int_{\mathbb{H}^n}\frac{u^p(y)v^q(y)}{|x^{-1}y|^\alpha|y|^\beta}dy, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `littleli_chen@163.com` (X. Chen), `xiaowei19901207@126.com` (W. Chen)

for $x\in\mathbb{H}^n$, where $0<\alpha<Q=2n+2$, $n\geq 3$, $\beta\geq 0$, $\alpha+\beta<Q$, and $p,q,r>1$ satisfying

$$\frac{1}{p+1}+\frac{1}{q+1}+\frac{1}{r+1}=\frac{Q+\alpha+\beta}{Q}. \tag{1.2}$$

The integral equation and integral equations in $\mathbb{R}^n$, have attracted a great deal of attentions and were studied by several different groups of people. With the method of moving planes in an integral form which was recently introduced by Chen, Li and Ou in [3], [4], the authors obtained all positive solutions of integral equations is radially symmetric and decreasing with respect to some point under integrable condition. Furthermore, the researchers also considered the other properties of integral system and using the method of lifting method [1], obtained the optimal integral interval and the asymptotic behavior. The reader can consult [2–5, 11, 12, 14] among numerous references, for its further development and applications.

The Hardy-Littlewood-Sobolev (HLS) inequality on the Heisenberg group $\mathbb{H}^n$ was further developed by Folland and Stein in [7] and proved that for all functions $f\in L^r(\mathbb{H}^n)$, $g\in L^s(\mathbb{H}^n)$, the following inequality

$$\left|\int\int_{\mathbb{H}^n\times\mathbb{H}^n}\frac{\overline{f(x)}g(y)}{|x^{-1}y|^\alpha}dxdy\right|\leq C_{\alpha,r,s,n}\|f\|_r\|g\|_s. \tag{1.3}$$

In conjunction with the CR Yamabe problem on the CR-manifolds, Jerison and Lee [10] showed the existence of sharp constant of Hardy-Littlewood-Sobolev inequality in the Heisenberg group $\mathbb{H}^n$, what's more, under some special conditions, i.e., $\alpha=Q-2$ and $r=s=\frac{2Q}{2Q-\alpha}$, the authors also obtain the explicit form of the constant sharp $C_{\alpha,r,s,n}$ and corresponding maximizing pair $f=g$. Frank and Lieb [8] extend the result of Jerison and Lee [10] to all $0<\alpha<Q$. Recently, Han, Lu and Zhu [9] generalized the HLS inequality (1.3) to the double weighted case, that is the Stein-Weiss inequalities or the double weighted HLS inequality on the Heisenberg group $\mathbb{H}^n$. Furthermore, using this inequality, they investigate the regularity, optimal integral interval and asymptotic behavior of positive solutions to integral systems

$$u(x)=\frac{1}{|x|^\tau}\int_{\mathbb{H}^n}\frac{v^q(y)}{|x^{-1}y|^\alpha|y|^\beta}dy,\quad v(x)=\frac{1}{|x|^\beta}\int_{\mathbb{H}^n}\frac{u^p(y)}{|x^{-1}y|^\alpha|y|^\tau}dy, \tag{1.4}$$

which are Euler-Lagrange equations of the possible extremals to the Stein-Weiss inequalities.

In this paper, we study system (1.1) with three components, similar problem in $\mathbb{R}^n$ was considered in [13]. Let $(u,v,w)\in L^{p+1}(\mathbb{H}^n)\times L^{q+1}(\mathbb{H}^n)\times L^{r+1}(\mathbb{H}^n)$ be any positive solution of (1.1). We assume condition (1.2) holds. Our first result asserts that such a solution is bounded.

Theorem 1.1. *Any positive solution triple* (u,v,w) *of* (1.1) *is bounded.*