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High Accuracy Spectral Method for the Space-Fractional Diffusion Equations

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Abstract. In this paper, a high order accurate spectral method is presented for the space-fractional diffusion equations. Based on Fourier spectral method in space and Chebyshev collocation method in time, three high order accuracy schemes are proposed. The main advantages of this method are that it yields a fully diagonal representation of the fractional operator, with increased accuracy and efficiency compared with low-order counterparts, and a completely straightforward extension to high spatial dimensions. Some numerical examples, including Allen-Cahn equation, are conducted to verify the effectiveness of this method.

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1 Introduction

Fractional differential equations have been proved to be valuable tools in modeling of many phenomena in various fields. In water resources, fractional models provide a useful description of chemical and contaminant transport in heterogeneous aquifers [1, 2]. In transport dynamics, they have been used to describe transport dynamics in complex

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systems which are governed by anomalous diffusion and non-exponential relaxation patterns [3]. Moreover, they are also used in finance, engineering and physics (see [4–6] and references cited therein).

In this paper, we consider the following space fractional diffusion equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -K(-\Delta)^{\alpha/2}u(x,t) + f(x,t,u), & (x,t)\in(a,b)\times(0,T], \\ u(x,0) = u_0(x), & x\in[a,b], \end{cases} \tag{1.1}$$

with the homogeneous Dirichlet or homogeneous Neumann boundary conditions. Here $K>0$ is the conductivity or diffusion tensor, and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian operator [7] with $1<\alpha<2$. The function $f=f(x,t,u)$ denotes the nonlinear source term.

There are many numerical methods to discretize the fractional Laplacian operator of problem (1.1). However, fractional differential operator is non-local red operator, which generates computational and numerical difficulties that have not been encountered in the context of the classical second-order diffusion equations. For space-fractional diffusion equations, numerical methods often generate full coefficient matrices with complicated structures [8–11]. In this paper we use Fourier spectral methods [12-14] to discretize the space-fractional derivative. This approach gives a full diagonal representation of the fractional operator and achieves spectral convergence regardless of the fractional power in the problem. Meanwhile, the application to high spatial dimensions is the same as the one-dimensional problem. For the temporal discretization, based on Chebyshev nodes [15, 16], the second-order Crank-Nicolson (CN) method and third-order implicit-explicit (IMEX) Runge-Kutta method [17] are used on the Chebyshev grids, respectively. Numerical experiments in Section 3 show that the time accuracy using Chebyshev grids is more accurate than using uniform grids.

The outline of this paper is as follows. In Section 2, three collocation/spectral numerical schemes are given for the space fractional diffusion equation (1.1). In Section 3, three numerical examples are carried out to verify the high efficiency of the proposed method, including the space-fractional Allen-Cahn equation in two dimensions. Finally, conclusions are drawn in Section 4.

2 High-order accurate schemes

In this section, we present three numerical schemes to simulate the asymptotic behavior of solution for the space fractional diffusion equation (1.1). The proposed schemes are based on Fourier spectral method in space and the collocation technique in time. In order to simplify the notations and without lose of generality, we only present numerical schemes for the one-dimensional space-fractional diffusion equation.