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Landau-Type Theorems for Solutions of a Quasilinear Differential Equation

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Abstract. In this paper, we study solutions of the quasilinear differential equation $\bar{z}\partial_{\bar{z}}f(z)+z\partial_z f(z)+(1-|z|^2)\partial_z\partial_{\bar{z}}f(z)=f(z)$. We utilize harmonic mappings to obtain an explicit representation of solutions of this equation. By this result, we give two versions of Landau-type theorem under proper normalization conditions.

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1 Introduction

Let $f(z)=u(x,y)+iv(x,y)$ be a twice continuously differentiable function of the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. If f satisfies the Laplacian equation $\Delta f(z)=4f_{z\bar{z}}=0$, then it is said to be a *harmonic mapping*. A harmonic mapping defined on a simply connected domain has a canonical expression

$$f(z)=h(z)+\overline{g(z)},$$

where $h(z)$ and $g(z)$ are analytic on $\mathbb{D}$. If a four times continuously differentiable function f satisfies $\Delta\Delta f(z)=0$, then it is said to be a *biharmonic mapping*, which has a representation with

$$f(z)=|z|^2p(z)+q(z),$$

where $p(z)$ and $q(z)$ are harmonic mappings of $\mathbb{D}$ (see [2]).

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For a continuously differentiable function $f(z), z\in\mathbb{D}$, we write

$$\begin{aligned}\Lambda_f(z) &= |f_z(z)|+|f_{\bar z}(z)|,\\ \lambda_f(z) &= ||f_z(z)|-|f_{\bar z}(z)||,\\ J_f(z) &= |f_z|^2-|f_{\bar z}|^2.\end{aligned}$$

Lewy [5] showed that a harmonic mapping is locally univalent if and only if its Jacobian $J_f(z)$ does not vanish for any $z\in\mathbb{D}$.

If $f(z)$ is a harmonic mapping of $\mathbb{D}$ satisfying that $\lim_{r\to 1} f(re^{i\theta}) = f^*(e^{i\theta})$ and $f^*(e^{i\theta})$ is a Lebesgue integrable function on $\mathbb{T}=\{z\in\mathbb{C}: |z|=1\}$, then

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi} P(ze^{-it})f^*(e^{it})dt,\ \ z\in\mathbb{D},$$

where $P(z)=\frac{1-|z|^2}{|1-z|^2}$ is the Poisson kernel of $\mathbb{D}$ (see [6, p.11]).

Define a kernel function $K(z)$ by

$$K(z)=\frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^4}. \tag{1.1}$$

Olofsson [13] introduced a quasilinear differential equation

$$\bar z\partial_{\bar z}f(z)+z\partial_z f(z)+(1-|z|^2)\partial_z\partial_{\bar z}f(z)=f(z), \tag{1.2}$$

and proved

Theorem A. *Suppose that* $f(z)\in C^2(\mathbb{D})$ *satisfies Eq.* (1.2) *and* $\lim_{r\to 1} f(re^{i\theta})=f^*(e^{i\theta})$, $z=re^{i\theta}$. *If* $f^*(e^{i\theta})$ *is a Lebesgue integrable function on* $\mathbb{T}$, *then*

$$f(z)=\frac{1}{2\pi}\int_{\mathbb{T}} K(ze^{-it})f^*(e^{it})dt,\ \ z\in\mathbb{D}. \tag{1.3}$$

In this paper, we show that the kernel function $K(z)$ is a biharmonic mapping, which is also a solution of Eq. (1.2) (see Lemma 2.1 in Section 2). Moreover, we utilize harmonic mappings to give an explicit representation of the solution of the Eq. (1.2) (see Theorem 2.1 in Section 2).

The classical Landau's theorem [9] states that if $f(z)$ is an analytic function on $\mathbb{D}$ satisfying that $f(0)=f'(0)-1=0$ and $|f(z)|\le M$ for $z\in\mathbb{D}$, then f is univalent in the disk $\mathbb{D}_{r_0}=\{z:|z|\le r_0\}$ with $r_0=\frac{1}{M+\sqrt{M^2-1}}$, and $f(\mathbb{D}_{r_0})$ contains a disk $\mathbb{D}_{\sigma_0}$, with $\sigma_0=Mr_0^2$. This result is sharp for the function $f(z)=Mz\frac{1-Mz}{M-z}$.

Recently, Landau's theorem has been introduced in other classes of mappings, Chen, Gauthier and Hengartner [3] obtained two versions of Landau-type theorems for bounded harmonic mappings. Under different normalization conditions, those papers [8, 10, 14] improved some results in [3]. Abdulhadi and Muhanna [1] gave two versions of Landau-type theorems for biharmonic mappings, and then their results were improved by the papers [4, 11, 12, 14]. Among them, Zhu and Liu [14] obtained

Theorem B. *Suppose that* $f(z)=|z|^2g(z)+h(z)$ *is a biharmonic mapping in the unit disk* $\mathbb{D}$ *such that* $|J_f(0)|=1$, $|g(z)|\le M_1$ *and* $|h(z)|\le M_2$ *for all* $z\in\mathbb{D}$.

(1) *If* $M_2>1$, *or* $M_2=1$ *and* $M_1>0$, *then* f *is univalent in the disk* $\mathbb{D}_{r_1}$, *and* $f(\mathbb{D}_{r_1})$ *contains a schlicht disk* $\mathbb{D}_{\sigma_1}=\{z\in\mathbb{D}:|z-f(0)|\le\sigma_1\}$, *where* $r_1=r_1(M_1,M_2)$ *is the minimum of positive roots of the following equation:*

$$\lambda_0(M_2)-2M_1r-\frac{4M_1r^2}{\pi(1-r^2)}-\lambda_0(M_2)\sqrt{M_2^4-1}\frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{3/2}}=0,$$

and

$$\sigma_1=\lambda_0(M_2)r_1-M_1r_1^2-\lambda_0(M_2)\sqrt{M_2^4-1}\frac{r_1^2}{(1-r_1^2)^{1/2}},$$

where $\lambda_0(M_2)$ *is given by* (3.2).

(2) *If* $M_2=1$*and* $M_1=0$, *then* f *is univalent in* $\mathbb{D}$ *and* $f(\mathbb{D})=\mathbb{D}$.

Theorem C. *Suppose that* $f(z)=|z|^2g(z)+h(z)$ *is a biharmonic mapping of the unit disk* $\mathbb{D}$, *such that* $\lambda_f(0)=1$, $|g(z)|\le M_1$ *and* $|h(z)|\le M_2$ *for* $z\in\mathbb{D}$.

(1) *If* $M_2>1$, *or* $M_2=1$ *and* $M_1>0$, *then* f *is univalent in the disk* $\mathbb{D}_{r_2}$, *and* $f(\mathbb{D}_{r_2})$ *contains a schlicht disk* $\mathbb{D}_{\sigma_2}=\{z\in\mathbb{C}:|z-f(0)|\le\sigma_2\}$, *where* $r_2=r_2(M_1,M_2)$ *is the minimum of positive roots of the following equation:*

$$1-2M_1r-\frac{4M_1r^2}{\pi(1-r^2)}-\sqrt{2(M_2^2-1)}\frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{3/2}}=0,$$

and

$$\sigma_2=r_2-M_1r_2^2-\sqrt{2(M_2^2-1)}\frac{r_2^2}{(1-r_2^2)^{1/2}}.$$

(2) *If* $M_2=1$ *and* $M_1=0$, *then* f *is univalent in* $\mathbb{D}$ *and* $f(\mathbb{D})=\mathbb{D}$.

In this paper, utilizing the representation (2.2) at Theorem 2.1, we obtain two versions of Landau-type theorem under the analogous normalization conditions given at the above Theorem B and Theorem C, correspondingly.

Theorem 1.1. *Let* $g(z)$ *be a harmonic mapping in* $\mathbb{D}$. *Suppose* $f(z)$ *is a solution of* (1.2) *with a representation*

$$f(z)=\frac{1}{2}(1-|z|^2)(zg_z(z)+\bar{z}g_{\bar{z}}(z))+\frac{1}{2}(1+|z|^2)g(z).$$

If $g(0)=|J_g(0)|-1=0$ *and* $|g(z)|\le M(M\ge1)$ *for all* $z\in\mathbb{D}$, *then* $f(z)$ *is univalent in the disk* $\mathbb{D}_{\rho_1}$ *and* $f(\mathbb{D}_{\rho_1})$ *contains a disk* $\mathbb{D}_{R_1}$, *where* ρ_1 *is the minimum of positive roots of the equation*

$\phi(\rho)=0$, $R_1=\psi(\rho_1)$, and

$$\phi(\rho)=\lambda_0(M)\left(1-\sqrt{M^4-1}\left(\frac{\rho\sqrt{\rho^4+4\rho^2+1}}{(1-\rho^2)^{5/2}}+\frac{\rho\sqrt{\rho^4-3\rho^2+4}}{(1-\rho^2)^{3/2}}\right)\right)$$
$$-\frac{4M\rho^2}{\pi(1-\rho^2)}-M\rho, \tag{1.4}$$
$$\psi(\rho)=\lambda_0(M)\rho-\frac{1}{2}\lambda_0(M)\sqrt{M^4-1}\frac{\rho^2(\sqrt{4\rho^4-11\rho^2+9}+\sqrt{\rho^2+1})}{(1-\rho^2)^{3/2}}, \tag{1.5}$$

where $\lambda_0(M)$ is given by (3.2).

Theorem 1.2. *Let $g(z)$ be a harmonic mapping in $\mathbb{D}$. Suppose $f(z)$ is a solution of* (1.2) *with a representation*

$$f(z)=\frac{1}{2}(1-|z|^2)(zg_z(z)+\bar{z}g_{\bar{z}}(z))+\frac{1}{2}(1+|z|^2)g(z).$$

If $g(0)=\lambda_g(0)-1=0$ and $|g(z)|\leq M(M\geq 1)$ for all $z\in\mathbb{D}$, then $f(z)$ is univalent in the disk $\mathbb{D}_{\rho_2}$ and $f(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{R_2}$, where ρ_2 is the minimum of positive roots of the following equation,

$$1-\sqrt{2(M^2-1)}\frac{\rho\sqrt{\rho^4+4\rho^2+1}}{(1-\rho^2)^{5/2}}-\sqrt{2(M^2-1)}\frac{\rho\sqrt{\rho^4-3\rho^2+4}}{(1-\rho^2)^{3/2}}-\frac{4M\rho^2}{\pi(1-\rho^2)}-M\rho=0,$$

and

$$R_2=\rho_2-\frac{1}{2}\sqrt{2(M^2-1)}\frac{\rho_2^2\left(\sqrt{4\rho_2^4-11\rho_2^2+9}+\sqrt{\rho_2^2+1}\right)}{(1-\rho_2^2)^{3/2}}.$$

2 Explicit representation of solutions of the quasilinear differential equation

In order to give a representation of a solution of the quasilinear differential equation (1.2), we first give some properties of the kernel function $K(z)$.

Lemma 2.1. *The kernel function $K(z)=\frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^4}$, $z\in\mathbb{D}$, is a solution of Eq.* (1.2) *and is also a biharmonic mapping.*

Proof. By some elementary calculations, we get

$$\partial_{\bar{z}}K(z)=\frac{1}{2}\Big(-3z(1-|z|^2)^2\frac{1}{|1-z|^4}+2(1-z)(1-|z|^2)^3\frac{1}{|1-z|^6}\Big)$$
$$=\frac{1}{2}\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-z+2\frac{1-z}{1-\bar{z}}\Big),$$

$$\partial_z K(z)=\frac{1}{2}\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-\bar{z}+2\frac{1-\bar{z}}{1-z}\Big),$$
$$\partial_z\partial_{\bar{z}}K(z)=\frac{1}{2}\frac{1-|z|^2}{|1-z|^4}\Big(-2z\frac{1-\bar{z}}{1-z}+4\Big)+\frac{1}{2}\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-1-\frac{2}{1-\bar{z}}\Big).$$

Using the above three relations, we have

$$\begin{aligned}
&\bar{z}\partial_{\bar{z}}K(z)+z\partial_z K(z)+(1-|z|^2)\partial_z\partial_{\bar{z}}K(z)\\
=&\frac{1}{2}\bar{z}\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-z+2\frac{1-z}{1-\bar{z}}\Big)+\frac{1}{2}z\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-\bar{z}+2\frac{1-\bar{z}}{1-z}\Big)\\
&+\frac{1}{2}(1-|z|^2)\frac{1-|z|^2}{|1-z|^4}\Big(-2z\frac{1-\bar{z}}{1-z}+4\Big)+\frac{1}{2}(1-|z|^2)\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-1-\frac{2}{1-\bar{z}}\Big)\\
=&\frac{1}{2}\frac{(1-|z|^2)^2}{|1-z|^4}\Big(-2|z|^2+2\bar{z}\frac{1-z}{1-\bar{z}}+4-(1-|z|^2)-2\frac{1-|z|^2}{1-\bar{z}}\Big)\\
=&\frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^4}=K(z).
\end{aligned}$$

That is, $K(z)$ is a solution of Eq. (1.2).

Since

$$K(z)=-\frac{1}{2}(1-|z|^2)+\frac{1}{2}\frac{1}{(1-z)^2}+\frac{1}{2}\frac{1}{(1-\bar{z})^2}-\frac{1}{2}\frac{z^2|z|^2}{(1-z)^2}-\frac{1}{2}\frac{\bar{z}^2|z|^2}{(1-\bar{z})^2}, \tag{2.1}$$

one shows that $K(z)$ is a biharmonic mapping, this completes the proof. □

Next we give an explicit representation of solution of Eq. (1.2).

Theorem 2.1. *Suppose that $f(z)\in C^2(\mathbb{D})$, satisfies $\lim_{r\to 1^-}f(re^{i\theta})=f^*(e^{i\theta})$ with $z=re^{i\theta}$. If $f^*(e^{i\theta})$ is a Lebesgue integrable function on the unit circle $\mathbb{T}$, then a solution of Eq.* (1.2) *has a representation*

$$f(z)=\frac{1}{2}(1-|z|^2)(zg_z(z)+\bar{z}g_{\bar{z}}(z))+\frac{1}{2}(1+|z|^2)g(z),\ \ z\in\mathbb{D}, \tag{2.2}$$

where $g(z)$ is a harmonic mapping in $\mathbb{D}$.

Proof. By (2.1), we have

$$K(z)=\frac{1}{2}\sum_{n=0}^{\infty}[(n+1)-(n-1)|z|^2]z^n+\frac{1}{2}\sum_{n=1}^{\infty}[(n+1)-(n-1)|z|^2]\bar{z}^n. \tag{2.3}$$

From Theorem A and the relation (2.3), we obtain

$$\begin{aligned}f(z)=&\frac{1}{2\pi}\int_{\mathbb{T}}K(ze^{-it})f^*(e^{it})dt\\=&\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1}{2}\sum_{n=0}^{\infty}[(n+1)-(n-1)|z|^2]z^ne^{-int}f^*(e^{it})dt\\&+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1}{2}\sum_{n=1}^{\infty}[(n+1)-(n-1)|z|^2]\bar{z}^ne^{int}f^*(e^{it})dt.\end{aligned}$$

Let $a_n=\frac{1}{2\pi}\int_{\mathbb{T}}f^*(e^{it})e^{-int}dt$, $\overline{b_n}=\frac{1}{2\pi}\int_{\mathbb{T}}f^*(e^{it})e^{int}dt$, $n=1,2,\cdots$. Thus, we have

$$f(z)=\frac{1}{2}\sum_{n=0}^{\infty}a_n((n+1)-(n-1)|z|^2)z^n+\frac{1}{2}\sum_{n=1}^{\infty}\overline{b_n}((n+1)-(n-1)|z|^2)\bar{z}^n.$$

Write $g(z)=\sum_{n=0}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}$. Then we get

$$f(z)=\frac{1}{2}(1-|z|^2)(zg_z(z)+\bar{z}g_{\bar{z}}(z))+\frac{1}{2}(1+|z|^2)g(z), \tag{2.4}$$

which shows that

$$f(z)=|z|^2\Big(\frac{1}{2}g(z)-\frac{1}{2}(zg_z(z)+\bar{z}g_{\bar{z}}(z))\Big)+\frac{1}{2}(g(z)+zg_z(z)+\bar{z}g_{\bar{z}}(z)).$$

Hence, $f(z)$ is a biharmonic mapping.

Next, we will prove that $f(z)$ are solutions of Eq. (1.2). Since

$$f_z=-\frac{1}{2}|z|^2g_z-\frac{1}{2}\bar{z}^2g_{\bar{z}}+\frac{1}{2}(1-|z|^2)zg_{zz}+\frac{1}{2}\bar{z}g+g_z, \tag{2.5}$$

$$f_{\bar{z}}=-\frac{1}{2}z^2g_z-\frac{1}{2}|z|^2g_{\bar{z}}+\frac{1}{2}(1-|z|^2)\bar{z}g_{\bar{z}\bar{z}}+\frac{1}{2}zg+g_{\bar{z}}, \tag{2.6}$$

we have

$$f_{\bar{z}z}=-\frac{1}{2}\bar{z}g_{\bar{z}}-\frac{1}{2}zg_z-\frac{1}{2}z^2g_{zz}-\frac{1}{2}\bar{z}^2g_{\bar{z}\bar{z}}+\frac{1}{2}g.$$

Thus,

$$\begin{aligned}&\bar{z}f_{\bar{z}}+zf_z+(1-|z|^2)f_{\bar{z}z}\\=&-\frac{1}{2}\bar{z}g_{\bar{z}}-\frac{1}{2}\bar{z}|z|^2g_{\bar{z}}-\frac{1}{2}zg_z-\frac{1}{2}z|z|^2g_z\\&+|z|^2g+zg_z+\bar{z}g_{\bar{z}}+\frac{1}{2}(1-|z|^2)g\\=&\frac{1}{2}(1-|z|^2)(zg_z(z)+\bar{z}g_{\bar{z}}(z))+\frac{1}{2}(1+|z|^2)g(z)\\=&f(z).\end{aligned}$$

This completes the proof. □

3 Proof of Theorem 1.1 and Theorem 1.2

In order to prove Theorem 1.1 and Theorem 1.2, we need the following lemmas.

Lemma A. [4,7] *Suppose that $f(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ such that $|f(z)|\leq M$ for all $z\in\mathbb{D}$. Then*

$$\Lambda_f(z)\leq\frac{4M}{\pi(1-|z|^2)},\quad z\in\mathbb{D}. \tag{3.1}$$

Moreover, the inequality (3.1) *is sharp.*

Lemma B. [11] *Suppose that $f(z)=h(z)+\overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ such that $|f(z)|\leq M$ for all $z\in\mathbb{D}$ with $h(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=1}^{\infty}b_nz^n$. If $|J_f(0)|=1$, then*

$$\lambda_f(0)\geq\lambda_0(M)=\begin{cases}\dfrac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}}, & \text{if } 1\leq M\leq M_0=\dfrac{\pi}{2\sqrt[4]{2\pi^2-16}},\\[2ex] \dfrac{\pi}{4M}, & \text{if } M\geq M_0=\dfrac{\pi}{2\sqrt[4]{2M^2-16}}\approx 1.1296.\end{cases} \tag{3.2}$$

Lemma C. [14] *Suppose that $f(z)=h(z)+\overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$, with $h(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=1}^{\infty}b_nz^n$.*

(1) *If f satisfies $|f(z)|\leq M$ for all $z\in\mathbb{D}$ and $|J_f(0)|=1$, then*

$$\left(\sum_{n=2}^{\infty}(|a_n|+|b_n|)^2\right)^{1/2}\leq\lambda_f(0)\cdot\sqrt{M^4-1}. \tag{3.3}$$

(2) *If f satisfies $|f(z)|\leq M$ for all $z\in\mathbb{D}$ and $\lambda_f(0)=1$, then*

$$\left(\sum_{n=2}^{\infty}(|a_n|+|b_n|)^2\right)^{1/2}\leq\sqrt{2(M^2-1)}. \tag{3.4}$$

Next, we will give prove the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Since $g(z)$ is a harmonic mapping in the unit disk, $g(z)$ can be written as $g(z)=h_1(z)+\overline{g_1}(z)$ with

$$h_1=\sum_{n=1}^{\infty}a_nz^n,\quad g_1=\sum_{n=1}^{\infty}b_nz^n,$$

where $h_1(z)$ and $g_1(z)$ are analytic in $\mathbb{D}$. By (2.5), we have

$$f_z=-\frac{1}{2}|z|^2h_1'-\bar{z}^2\overline{g_1'}+\frac{1}{2}(1-|z|^2)zh_1''+\frac{1}{2}\bar{z}g+h_1',$$

thus, $f_z(0)=g_z(0)=h_1'(0)$. By (2.6), we also get

$$f_{\bar{z}}=-\frac{1}{2}z^2h_1'-\frac{1}{2}|z|^2\overline{g_1'}+\frac{1}{2}(1-|z|^2)\bar{z}\overline{g_1''}+\frac{1}{2}zg+\overline{g_1'},$$

thus, $f_{\bar{z}}(0)=g_{\bar{z}}(0)=\overline{g_1'(0)}$.

For fixed $\rho\in(0,1)$, we choose z_1,z_2 with $z_1\neq z_2$, $|z_1|<\rho$ and $|z_1|<\rho$. Let $[z_1,z_2]$ be the line segment from z_1 to z_2. Then, it follows that

$$\begin{aligned}|f(z_2)-f(z_1)|&=\left|\int_{[z_1,z_2]}f_z(z)dz+f_{\bar{z}}(z)d\bar{z}\right|\\&\geq\left|\int_{[z_1,z_2]}f_z(0)dz+f_{\bar{z}}(0)d\bar{z}\right|\\&\quad-\left|\int_{[z_1,z_2]}(f_z(z)-f_z(0))dz+(f_{\bar{z}}(z)-f_{\bar{z}}(0))d\bar{z}\right|.\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{[z_1,z_2]}f_z(0)dz+f_{\bar{z}}(0)d\bar{z}\right|\\=&\int_{[z_1,z_2]}|f_z(0)+f_{\bar{z}}(0)e^{-2i\theta}|ds\geq\lambda_f(0)|z_1-z_2|\\=&\lambda_g(0)|z_1-z_2|.\end{aligned}$$

Using Lemma A and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left|\int_{[z_1,z_2]}(f_z(z)-f_z(0))dz+(f_{\bar{z}}(z)-f_{\bar{z}}(0))d\bar{z}\right|\\\leq&\frac{1}{2}\left|\int_{[z_1,z_2]}\bar{z}^2\bar{g}_1'dz+z^2h_1'd\bar{z}\right|+\frac{1}{2}\left|\int_{[z_1,z_2]}|z|^2h_1'dz+|z|^2\overline{g_1'}d\bar{z}\right|\\&+\frac{1}{2}\left|\int_{[z_1,z_2]}(1-|z|^2)zh_1''dz+(1-|z|^2)\bar{z}\overline{g_1''}d\bar{z}\right|+\frac{1}{2}\left|\int_{[z_1,z_2]}g(\bar{z}dz+zd\bar{z})\right|\\&+\left|\int_{[z_1,z_2]}(h_1'-h_1'(0))dz+(\overline{g_1}'-\overline{g_1'(0)})d\bar{z}\right|\\\leq&|z_1-z_2|\left\{\frac{4M\rho^2}{\pi(1-\rho^2)}+\frac{1}{2}\sum_{n=2}^{\infty}(|a_n|+|b_n|)n(n-1)\rho^{n-1}\right.\\&\left.+M\rho+\sum_{n=2}^{\infty}(|a_n|+|b_n|)n\rho^{n-1}\right\}\\\leq&|z_1-z_2|\left\{\frac{4M\rho^2}{\pi(1-\rho^2)}+\frac{1}{2}\sum_{n=2}^{\infty}(|a_n|+|b_n|)^2)^{1/2}\left(\sum_{n=2}^{\infty}n^2(n-1)^2\rho^{2(n-1)}\right)^{1/2}\right.\\&\left.+M\rho+\left(\sum_{n=2}^{\infty}(|a_n|+|b_n|)^2\right)^{1/2}\left(\sum_{n=2}^{\infty}n^2\rho^{2(n-1)}\right)^{1/2}\right\}.\end{aligned}$$

The combination of Lemma B with Lemma C gives us that

$$\begin{aligned}
&|f(z_2)-f(z_1)|\\
\geq& |z_1-z_2|\Big\{\lambda_g(0)-\frac{4M\rho^2}{\pi(1-\rho^2)}-\frac{1}{2}\lambda_g(0)\sqrt{M^4-1}\frac{2\rho\sqrt{\rho^4+4\rho^2+1}}{(1-\rho^2)^{5/2}}\\
&-M\rho-\lambda_g(0)\sqrt{M^4-1}\frac{\rho\sqrt{\rho^4-3\rho^2+4}}{(1-\rho^2)^{3/2}}\Big\}\\
\geq& |z_1-z_2|\Big\{\lambda_0(M)-\frac{4M\rho^2}{\pi(1-\rho^2)}-\lambda_0(M)\sqrt{M^4-1}\frac{\rho\sqrt{\rho^4+4\rho^2+1}}{(1-\rho^2)^{5/2}}\\
&-M\rho-\lambda_0(M)\sqrt{M^4-1}\frac{\rho\sqrt{\rho^4-3\rho^2+4}}{(1-\rho^2)^{3/2}}\Big\},
\end{aligned}$$

this implies that there exists a ρ_1 satisfying $|f(z_2)-f(z_1)|>0$. For any $|z'|=\rho_1$, we have

$$\begin{aligned}
|f(z')|=&\frac{1}{2}|(1-|z'|^2)(z'g_z(z')+\bar{z}'g_{\bar{z}}(z')+(1+|z'|^2)g(z')|\\
\geq& |(a_1z'+\overline{b_1z'})|-\frac{1}{2}\Big|\sum_{n=2}^{\infty}(n+1)a_nz'^n+\overline{\sum_{n=2}^{\infty}(n+1)b_nz'^n}\Big|\\
&-\frac{1}{2}|z|^2\Big|\sum_{n=2}^{\infty}(n-1)a_nz'^n+\overline{\sum_{n=2}^{\infty}(n-1)b_nz'^n}\Big|\\
\geq& \lambda_0(M)\rho_1-\frac{1}{2}\lambda_0(M)\sqrt{M^4-1}\frac{\rho_1^2\left(\sqrt{4\rho_1^4-11\rho_1^2+9}+\sqrt{\rho_1^2+1}\right)}{(1-\rho_1^2)^{3/2}},
\end{aligned}$$

this completes the proof. □

Remark 3.1. For any $\rho\in(0,1)$, $\phi(\rho)$ is a continuous function. Since $\lim_{\rho\to0^+}\phi(\rho)=\lambda_0(M)$ and $\lim_{\rho\to1^-}\phi(\rho)=-\infty$, the equation $\phi(\rho)$ has a root in $(0,1)$. Since $R_1=\psi(\rho_1)>\rho_1\phi(\rho_1)$, we have $R_1>0$.

Proof of Theorem 1.2. Notice that $\lambda_f(0)=1$. We replace $\lambda_0(M)$ in (1.4) and (1.5) with $\lambda_f(0)$. Using the same method as the proof of Theorem 1.1, we obtain the proof of Theorem 1.2 from the relation (3.4) at Lemma C. □

Remark 3.2. Since $\sqrt{2(M^2-1)}\leq\sqrt{M^4-1}$, we have $\rho_1<\rho_2$, $R_1<R_2$ at Theorem 1.1 and Theorem 1.2.

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