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Landau-Type Theorems for Solutions of a Quasilinear Differential Equation

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Abstract. In this paper, we study solutions of the quasilinear differential equation $\bar{z}\partial_{\bar{z}}f(z)+z\partial_z f(z)+(1-|z|^2)\partial_z\partial_{\bar{z}}f(z)=f(z)$. We utilize harmonic mappings to obtain an explicit representation of solutions of this equation. By this result, we give two versions of Landau-type theorem under proper normalization conditions.

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1 Introduction

Let $f(z)=u(x,y)+iv(x,y)$ be a twice continuously differentiable function of the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. If f satisfies the Laplacian equation $\Delta f(z)=4f_{z\bar{z}}=0$, then it is said to be a *harmonic mapping*. A harmonic mapping defined on a simply connected domain has a canonical expression

$$f(z)=h(z)+\overline{g(z)},$$

where $h(z)$ and $g(z)$ are analytic on $\mathbb{D}$. If a four times continuously differentiable function f satisfies $\Delta\Delta f(z)=0$, then it is said to be a *biharmonic mapping*, which has a representation with

$$f(z)=|z|^2p(z)+q(z),$$

where $p(z)$ and $q(z)$ are harmonic mappings of $\mathbb{D}$ (see [2]).

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For a continuously differentiable function $f(z), z\in\mathbb{D}$, we write

$$\Lambda_f(z)=|f_z(z)|+|f_{\bar z}(z)|,$$
$$\lambda_f(z)=||f_z(z)|-|f_{\bar z}(z)||,$$
$$J_f(z)=|f_z|^2-|f_{\bar z}|^2.$$

Lewy [5] showed that a harmonic mapping is locally univalent if and only if its Jacobian $J_f(z)$ does not vanish for any $z\in\mathbb{D}$.

If $f(z)$ is a harmonic mapping of $\mathbb{D}$ satisfying that $\lim_{r\to 1} f(re^{i\theta})=f^*(e^{i\theta})$ and $f^*(e^{i\theta})$ is a Lebesgue integrable function on $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$, then

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi} P(ze^{-it})f^*(e^{it})dt,\ \ z\in\mathbb{D},$$

where $P(z)=\frac{1-|z|^2}{|1-z|^2}$ is the Poisson kernel of $\mathbb{D}$ (see [6, p.11]).

Define a kernel function $K(z)$ by

$$K(z)=\frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^4}. \tag{1.1}$$

Olofsson [13] introduced a quasilinear differential equation

$$\bar z\partial_{\bar z} f(z)+z\partial_z f(z)+(1-|z|^2)\partial_z\partial_{\bar z} f(z)=f(z), \tag{1.2}$$

and proved

Theorem A. *Suppose that* $f(z)\in C^2(\mathbb{D})$ *satisfies Eq.* (1.2) *and* $\lim_{r\to 1} f(re^{i\theta})=f^*(e^{i\theta})$, $z=re^{i\theta}$. *If* $f^*(e^{i\theta})$ *is a Lebesgue integrable function on* $\mathbb{T}$, *then*

$$f(z)=\frac{1}{2\pi}\int_{\mathbb{T}} K(ze^{-it})f^*(e^{it})dt,\ \ z\in\mathbb{D}. \tag{1.3}$$

In this paper, we show that the kernel function $K(z)$ is a biharmonic mapping, which is also a solution of Eq. (1.2) (see Lemma 2.1 in Section 2). Moreover, we utilize harmonic mappings to give an explicit representation of the solution of the Eq. (1.2) (see Theorem 2.1 in Section 2).

The classical Landau's theorem [9] states that if $f(z)$ is an analytic function on $\mathbb{D}$ satisfying that $f(0)=f'(0)-1=0$ and $|f(z)|\le M$ for $z\in\mathbb{D}$, then f is univalent in the disk $\mathbb{D}_{r_0}=\{z:|z|\le r_0\}$ with $r_0=\frac{1}{M+\sqrt{M^2-1}}$, and $f(\mathbb{D}_{r_0})$ contains a disk $\mathbb{D}_{\sigma_0}$, with $\sigma_0=Mr_0^2$. This result is sharp for the function $f(z)=Mz\frac{1-Mz}{M-z}$.

Recently, Landau's theorem has been introduced in other classes of mappings, Chen, Gauthier and Hengartner [3] obtained two versions of Landau-type theorems for bounded harmonic mappings. Under different normalization conditions, those papers [8, 10, 14] improved some results in [3]. Abdulhadi and Muhanna [1] gave two versions of Landau-type theorems for biharmonic mappings, and then their results were improved by the papers [4, 11, 12, 14]. Among them, Zhu and Liu [14] obtained