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Properties of Convergence for a Class of Generalized q-Gamma Operators

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Abstract. In this paper, a generalization of q-Gamma operators based on the concept of q-integer is introduced. We investigate the moments and central moments of the operators by computation, obtain a local approximation theorem and get the pointwise convergence rate theorem and also obtain a weighted approximation theorem. Finally, a Voronovskaya type asymptotic formula was given.

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1 Introduction

It is well known that the Gamma operators are given by

$$G_n(f;x)=\frac{1}{x^n\Gamma(n)}\int_0^\infty f(t/n)t^{n-1}e^{-t/x}dt,\ \ x\in[0,\infty). \tag{1.1}$$

In 2005, Zeng [9] obtained the approximation properties of G_n defined above, supposing f satisfies exponential growth condition. He studied the approximation properties to the locally bounded functions and the absolutely continuous functions and obtained some good properties.

Since the application of q-calculus in approximation theory is an active field, many researchers have performed studies in it, we mention some of them [3,5–8], these motivate us to introduce the q analogue of this kind of Gamma operators.

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Firstly, we recall some concepts of q-calculus. All of the results can be found in [4]. For any fixed real number $0<q\leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q=\begin{cases} \frac{1-q^k}{1-q}, & q\neq 1; \\ k, & q=1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q!=\begin{cases} [k]_q[k-1]_q...[1]_q, & k=1,2,...; \\ 1, & k=0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n\geq k\geq 0).$$

The q-improper integrals are defined as

$$\int_0^{\infty/A} f(x)d_qx=(1-q)\sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \ A>0, \tag{1.2}$$

provided the sums converge absolutely.

The q-exponential function $E_q(x)$ is given as

$$E_q(x)=\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]_q!}=(1+(1-q)x)_q^{\infty}, \ |q|<1,$$

where $(1-x)_q^{\infty}=\prod_{j=0}^{\infty}(1-q^jx)$.

The q-Gamma integral is defined as

$$\Gamma_q(t)=\int_0^{\infty/A} x^{t-1}E_q(-qx)d_qx, \ t>0, \tag{1.3}$$

which satisfies the following functional equations: $\Gamma_q(t+1)=[t]_q\Gamma_q(t), \ \Gamma_q(1)=1$.

For $f\in C[0,\infty)$, $q\in(0,1)$ and $n\in\mathbb{N}$, we introduce a generalization of q-Gamma operators $G_{n,q}(f,x)$ as

$$G_{n,q}(f;x)=\frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/A} f\left(\frac{t}{[n]_q}\right)t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt. \tag{1.4}$$

Obviously, $G_{n,q}(f;x)$ are positive linear operators. It is observed that for $q\to 1^-$, $G_{n,1^-}(f;x)$ become the Gamma operators defined in (1.1).