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A Comparative Numerical Study of Meshing Functionals for Variational Mesh Adaptation

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Abstract. We present a comparative numerical study for three functionals used for variational mesh adaptation. One of them is a generalization of Winslow's variable diffusion functional while the others are based on equidistribution and alignment. These functionals are known to have nice theoretical properties and work well for most mesh adaptation problems either as a stand-alone variational method or combined within the moving mesh framework. Their performance is investigated numerically in terms of equidistribution and alignment mesh quality measures. Numerical results in 2D and 3D are presented.

AMS subject classifications: 65N50, 65K10

Key words: Variational mesh adaptation, Mesh adaptation, Moving mesh, Equidistribution, Alignment, Mesh quality measures.

1 Introduction

Variational mesh adaptation is an important type of mesh adaptation method and has received considerable attention from scientists and engineers; e.g., see the books [15, 19, 23, 24] and references therein. It also serves as the base of a number of commonly used adaptive moving mesh methods (e.g., see [5, 12, 14, 22]). In the variational approach, an adaptive mesh is generated as the image of a reference mesh under a coordinate transformation and such a coordinate transformation is determined as a minimizer of a certain meshing functional. A number of meshing functionals have been developed in the past (cf. the above mentioned books). For example, Winslow [25] proposed an equipotential method based on variable diffusion. Brackbill and Saltzman [3] developed a method by

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combining mesh concentration, smoothness, and orthogonality. Dvinsky [6] used the energy of harmonic mappings as his meshing functional, while Knupp [20] and Knupp and Robidoux [21] developed functionals based on the idea of conditioning the Jacobian matrix of the coordinate transformation. More recently, Huang [7] and Huang and Russell [15] proposed functionals based on the so-called equidistribution and alignment conditions.

With variational mesh adaptation, the mesh concentration is typically controlled through a scalar or a matrix-valued function, often referred to as the metric tensor or monitor function and defined based on some error estimates and/or physical considerations. While most of the meshing functionals have been developed with physical or geometric intuitions and have various levels of success in the adaptive numerical solution of partial differential equations (PDEs) and other applications, there is only a limited understanding on how the metric tensor affects the behavior of the mesh. An attempt to alleviate this lack of understanding was made by Cao *et al.* [4] for a generalization of Winslow's variable diffusion functional. They showed that a significant change in an eigenvalue of the metric tensor along the corresponding eigendirection (first increasing and then decreasing, or vice versa) will result in adaptation of coordinate lines along this direction, although this adaptation competes with far more complicated effects, including those from changes in eigenvectors and other eigenvalues and the effects of the shapes of the physical and computational domains and the mesh point distribution on the boundaries. In [7,15], two functionals have been developed based directly on the equidistribution and alignment conditions. These two conditions provide a complete characterization of the mesh elements through the metric tensor [7]. Minimizing the functionals leads to meshes which tend to satisfy the conditions in an integral sense. Nevertheless, this characterization is only qualitative, and how closely the mesh satisfies the conditions depends on the boundary correspondence between the computational and physical domains and the mesh point distribution on the boundaries. Thus, numerical studies, especially comparative ones, are useful, and often necessary, in understanding how the mesh adaptation for those meshing functionals is controlled precisely by the metric tensor. There do exist a few comparative numerical studies for meshing functionals. For example, a gallery of (adaptive and non-adaptive) meshes is given in [19] for a number of meshing functionals. Some comparative meshes are given in [15] for the harmonic mapping functional [6] and the subsequent functional based on equidistribution and alignment [7].

The main objective of this work is to present a comparative study for three of the most appealing meshing functionals, a generalization of Winslow's variable diffusion functional (cf. (3.2)) and two functionals based on equidistribution and alignment (cf. (3.7) and (3.9)). They are selected because (3.2) and (3.7) have been known to work well for many problems (e.g., see [1, 2, 7, 13, 14, 22]) while (3.9) is similar to (3.7) and has some very nice theoretical properties (cf. §3.2). Another motivation is to present some three dimensional numerical results for those functionals. Critical for our study is to perform the substantial computations using the improved implementation of the variational methods introduced in [11]. In a sharp contrast to the situation in two dimensions, very little work