The Distortion Theorems for Harmonic Mappings with Negative Coefficient Analytic Parts

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Abstract. Some sharp estimates for coefficients, distortion and the growth order are obtained for harmonic mappings $f \in TL^\alpha H$ which are locally univalent harmonic mappings in the unit disk $D := \{z : |z| < 1\}$. Moreover, denoting the subclass $TS^\alpha H$ of the normalized univalent harmonic mappings, we also estimate the growth of $|f|, f \in TS^\alpha H$, and their covering theorems.

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1 Introduction

Let $S$ denote the class of functions of the form $F(z) = z + \sum_{n=2}^{\infty} a_n z^n$, that are analytic and univalent in the unit disk $D := \{z : |z| < 1\}$. Denoting $T$ to be the subclass of $S$ consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an univalent analytic function $F \in T$ if and only if it can be written in the form

$$F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in D. \quad (1.1)$$

A complex-valued harmonic function $f$ in the unit disk $D$ has a canonical decomposition

$$f(z) = h(z) + \overline{g(z)} \quad (1.2)$$

where $h$ and $g$ are analytic in $D$ with $g(0) = 0$. Usually, we call $h$ the analytic part of $f$ and $g$ the co-analytic part of $f$. A complete and elegant account of the theory of planar harmonic mappings is given in Duren’s monograph [1].

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In [2], Ikkei Hotta and Andrzej Michalski denoted the class $L_H$ of all normalized locally univalent and sense-preserving harmonic functions in the unit disk with $h(0) = g(0) = h'(0) - 1 = 0$. Which means every function $f \in L_H$ is uniquely determined by coefficients of the following power series expansions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \quad (1.3)$$

where $a_n, b_n \in \mathbb{C}, n = 2, 3, 4, \ldots$. Clunie and Sheil-small introduced in [3] the class $S_H$ of all normalized univalent harmonic mappings in $\mathbb{D}$, obviously, $S_H \subset L_H$.

Lewy [4] proved that a necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\mathbb{D}$ is

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2, \quad z \in \mathbb{D}. \quad (1.4)$$

To such a function $f$, not identically constant, let

$$\omega(z) = \frac{g'(z)}{h'(z)}, \quad z \in \mathbb{D}, \quad (1.5)$$

then $\omega(z)$ is analytic in $\mathbb{D}$ with $|\omega(z)| < 1$, it is called the second complex dilatation of $f$.

In [5], Silverman investigated the subclass of $T$ which denoted by $T^*(\beta)$, starlike of order $\beta(0 \leq \beta < 1)$. That is, a function $F(z) \in T^*(\beta)$ if $\Re\{zF'(z)/F(z)\} > \beta, \quad z \in \mathbb{D}$. It was proved in [5] that

**Corollary 1.1.**

$$T = T^*(0).$$

In [7-8], Dominika Klimek and Andrzej Michalski studied the cases when the analytic parts $h$ is the identity mapping or a convex mapping, respectively. The paper [2] was devoted to the case when the analytic $h$ is a starlike analytic mapping. In [9], Qin Deng got sharp results concerning coefficient estimate, distortion theorems and covering theorems for functions in $T$. The main idea of this paper is to characterize the subclasses of $L_H$ and $S_H$ when $h \in T$.

In order to establish our main results, we need the following theorems and lemmas.

**Theorem 1.1.** ([8]) A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $T$ if and only if

$$\sum_{n=2}^{\infty} n |a_n| \leq 1, \quad z \in \mathbb{D}. \quad (1.6)$$

**Lemma 1.1.** ([10]) If $f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$ is analytic and $|f(z)| \leq 1$ on $\mathbb{D}$, then

$$|a_n| \leq 1 - |a_0|^2, \quad n = 1, 2, \ldots. \quad (1.7)$$